Hyperbolic models for dispersive phenomena

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Personal experience

Each presentation is made up of original and unoriginal parts. Parts that are good are generally unoriginal, and parts that are not very good are original.

Linear hyperbolic and dispersive waves

Consider plane waves

$$\boldsymbol{U} = \boldsymbol{a} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$$

that are solutions of a system of linear PDEs

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having real dispersion relation $\omega = \omega(\boldsymbol{k})$.

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- The waves are hyperbolic if $\det||\omega''(\boldsymbol{k})|| = 0$.
- The waves are dispersive, if $\det||\omega''(\boldsymbol{k})|| \neq 0$.

Symmetric t-hyperbolic in the sense of Friedrichs systems

Symmetric t-hyperbolic in the sense of Friedrichs systems (Kurt Otto Friedrichs, 1901-1982)

$$\boldsymbol{A}_0 \boldsymbol{U}_t + \sum_s \boldsymbol{A}_s \boldsymbol{U}_{x_s} = 0, \quad \boldsymbol{A}_s = \boldsymbol{A}_s{}^T, \; \boldsymbol{A}_0 = \boldsymbol{A}_0{}^T > 0, \quad \boldsymbol{U} = (U_1, ..., U_n)^T.$$

The Cauchy problem is well posed (T. Kato, The Cauchy Problem for Quasi-Linear Symmetric Hyperbolic Systems, Arch. Rational Mech. Anal. 58, 181–205 (1975)).

Godunov-Lax-Friedrichs theorem

Godunov-Lax-Friedrichs theorem. If the system of conservation laws

$$\boldsymbol{V}_t + \sum_s \boldsymbol{F}_s(\boldsymbol{V})_{x_s} = 0 \qquad (1)$$

admits a convex conservation law

$$E(\boldsymbol{V})_t + \sum_s G_s(\boldsymbol{V})_{x_s} = 0 \qquad (2)$$

then the equations can be written as a symmetric t-hyperbolic system in the sense of Friedrichs.

Linear homogeneous hyperbolic equations

$$\boldsymbol{A}_0 \boldsymbol{U}_t + \sum_s \boldsymbol{A}_s \boldsymbol{U}_{x_s} = 0, \quad \boldsymbol{A}_s = {\boldsymbol{A}_s}^T, \ \boldsymbol{A}_0 = {\boldsymbol{A}_0}^T > 0.$$

Plane waves

$$\boldsymbol{U} = \boldsymbol{a} \exp^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$$

Dispersion relation

$$\det(\sum_s \boldsymbol{A}_s k_s - \omega \boldsymbol{A}_0) = 0.$$

$\omega = \omega(\boldsymbol{k})$ is a real and homogeneous function of $\boldsymbol{k}$ of power 1. It implies $\omega = \boldsymbol{k} \cdot \nabla\omega$, and finally $\omega''(\boldsymbol{k})\boldsymbol{k} = 0$. So, linear homogeneous hyperbolic systems of equations describe hyperbolic linear waves.

Dispersive waves described by linear hyperbolic equations

$$\boldsymbol{A}_0 \boldsymbol{U}_t + \sum_s \boldsymbol{A}_s \boldsymbol{U}_{x_s} = \boldsymbol{C}\boldsymbol{U}, \quad \boldsymbol{A}_s = {\boldsymbol{A}_s}^T,\ \boldsymbol{A}_0 = {\boldsymbol{A}_0}^T > 0,\ \boldsymbol{C}^T = -\boldsymbol{C}.$$

The dispersion relation is real (prove it !) and given by n roots of the dispersion relation

$$det(\sum_s \boldsymbol{A}_s k_s - \omega \boldsymbol{A}_0 + i\boldsymbol{C}) = 0.$$

Linear equations describing dispersive waves

Linear KdV equation

$$u_t + cu_x + u_{xxx} = 0$$

Linear equations describing dispersive waves

Linear KdV equation

$$u_t + cu_x + u_{xxx} = 0$$

Dispersion relation

$$\omega = ck - k^3$$

Attempt of hyperbolization

$$u_t + cu_x + \psi_x = 0, \quad p_t - \frac{1}{\varepsilon}(u_x - p) = 0, \quad \psi_t - \frac{1}{\varepsilon}(p_x - \psi) = 0.$$

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Characteristic polynomial :

$$\lambda^2(c - \lambda) + \frac{1}{\varepsilon^2} = 0.$$

The equations are not hyperbolic.

'Intuitive' hyperbolic approximation

$$u_t + cu_x + u_{xxx} = 0, \quad \left(\frac{u^2}{2}\right)_t + \left(c\frac{u^2}{2} + uu_{xx} - \frac{u_x^2}{2}\right)_x = 0$$

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Conservation law (almost H^2 norm)

$$\left(\frac{u^2}{2} + \varepsilon\frac{p^2}{2} + \varepsilon\frac{\psi^2}{2}\right)_t + \left(c\frac{u^2}{2} + u\psi - \frac{p^2}{2}\right)_x = 0$$

Remarks

- Could be written in the form

$$\boldsymbol{A}_0\boldsymbol{U}_t + \boldsymbol{A}\boldsymbol{U}_x = C\boldsymbol{U},$$

$$\boldsymbol{A}_0 = \begin{pmatrix} 1/\varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} c/\varepsilon & 0 & 1/\varepsilon \\ 0 & -1/\varepsilon & 0 \\ 1/\varepsilon & 0 & 0 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/\varepsilon \\ 0 & 1/\varepsilon & 0 \end{pmatrix}$$

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- It looks as a relaxation system in the sense of I. Suliciu, F. Couqel, B. Perthame, F. Bouchut, J. M. Herard, K. Saleh, ..., but it is not.

Whitham's condition

Conservation law

$$u_t + f(u)_x = 0.$$

Jin-Xin system

$$u_t + v_x = 0, \quad v_t + a^2 u_x = \frac{f(u) - v}{\varepsilon}, \quad |f'(u)| < a.$$

What are nonlinear dispersive waves ?

They can probably be defined within the framework of a certain structure.

Generic (nonclassical) definition of nonlinear dispersive waves in continuum mechanics

Definitions

- $E = T + W$ – total energy
- T – kinetic energy
- W – potential energy
- $L = T - W$ – Lagrangian
- $a = \int_{t_0}^{t_1} \int_{\mathcal{D}(t)} L dD dt$ - Hamilton's action

Hamilton's principle

The governing equations are stationary 'points' of Hamilton's action (under certain constraints to be defined).

$$L = \rho \left(\frac{|\boldsymbol{v}|^2}{2} - \tilde{e}(\eta, \boldsymbol{F}, \nabla \boldsymbol{F}, \dot{\boldsymbol{F}}) \right)$$

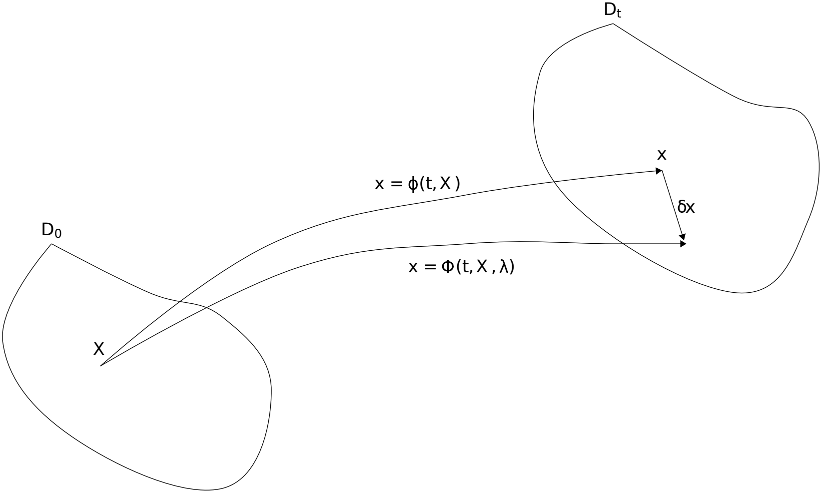
Here $\boldsymbol{F}$ is the deformation gradient, *dot* means the material derivative.

Constraints

The constraints should be integrable in the reference configuration !

- Conservation of the mass
- Conservation of the entropy
- ...

Motion and virtual motion



Motion and virtual motion

- $\boldsymbol{x} = \varphi(t, \boldsymbol{X}),\ \boldsymbol{x} = \left(x^1, x^2, x^3\right)^T,\ \boldsymbol{X} = \left(X^1, X^2, X^3\right)^T,$

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- $\boldsymbol{x} = \Phi(t, \boldsymbol{X}, \lambda)$,

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- $\boldsymbol{x} = \Phi(t, \boldsymbol{X}, \lambda)$,
- $\delta\boldsymbol{x}(t, \boldsymbol{X}) = \frac{\partial}{\partial\lambda}\Phi(t, \boldsymbol{X}, \lambda)|_{\lambda=0}$,
- $\zeta(t, \boldsymbol{x}) = \delta\boldsymbol{x}\left(t, \varphi^{-1}(t, \boldsymbol{x})\right)$.

Lagrangian and Eulerian variations

$$\tilde{f}(t, \boldsymbol{X}, \lambda), \;\; \hat{f}(t, \boldsymbol{x}, \lambda).$$

$$\tilde{\delta} f = \frac{\partial}{\partial \lambda} \tilde{f}(t, \boldsymbol{X}, \lambda)|_{\lambda=0}, \quad \hat{\delta} f = \frac{\partial}{\partial \lambda} \hat{f}(t, \boldsymbol{x}, \lambda)|_{\lambda=0},$$

$$\tilde{\delta} f = \hat{\delta} f + \nabla f \cdot \delta \boldsymbol{x}.$$

Lagrangian variations

- $\tilde{\delta}\rho = -\rho div(\zeta)$,
- $\tilde{\delta}\eta = 0, \ \tilde{\delta}\boldsymbol{F}^{-T} = -\left(\frac{\partial\zeta}{\partial\boldsymbol{x}}\right)^T \boldsymbol{F}^{-T}$,
- $\tilde{\delta}\boldsymbol{u} = \frac{\partial\delta\boldsymbol{x}}{\partial t}$.

Eulerian variations

- $\hat{\delta}\rho = -div(\rho\boldsymbol{\zeta})$,
- $\hat{\delta}\eta = -\nabla\eta \cdot \boldsymbol{\zeta}$, $\hat{\delta}\boldsymbol{F} = -\left(\frac{\partial\boldsymbol{\zeta}}{\partial\boldsymbol{x}}\right)^T \boldsymbol{F} - \left(\frac{\partial\boldsymbol{F}}{\partial\boldsymbol{x}}\right)\boldsymbol{\zeta}$,
- $\hat{\delta}\boldsymbol{u} = \frac{D\boldsymbol{\zeta}}{Dt} - \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}}\boldsymbol{\zeta}$.

How to obtain the equations of motion ?

$$\delta a = \int_{t_0}^{t_1} \int_{\boldsymbol{D}(t)} \boldsymbol{M} \cdot \zeta dD \, dt = 0.$$

It implies

$$\boldsymbol{M} = 0.$$

Euler equations of compressible fluids

- $T = \int_{\mathcal{D}_t} \rho \frac{\|\boldsymbol{u}\|^2}{2} dD$
- $W = \int_{\mathcal{D}_t} \rho\varepsilon\left(\rho, \eta\right) dD.$

Constraints

$$\rho_t + div(\rho\boldsymbol{u}) = 0, \quad (\rho\eta)_t + div(\rho\eta\boldsymbol{u}) = 0.$$

Equations

$$\left(\rho\boldsymbol{u}\right)_t + div\left(\rho\boldsymbol{u} \otimes \boldsymbol{u} + p\boldsymbol{I}\right) = 0.$$

The convexity of the specific energy $\varepsilon(\tau, \eta)$, $\tau = 1/\rho$ implies the hyperbolicity.

General framework for a class of dispersive fluid models

Generic stored potential :

$$\tilde{e}\left(\rho, \nabla\rho, \dot{\rho}\right)$$

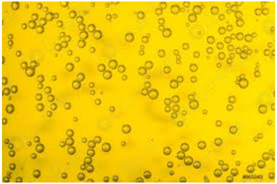
"Inertia " type dispersion (S. V. Iordansky, B. Kogarko, L. van Wijngaarden, SG and V. Teshukov, M. Massot, S. Kokh, ...)

$$\tilde{e}\left(\rho, \dot{\rho}\right)$$

"Capillarity" type dispersion (D. Korteweg, J. D. van der Waals, M. Eglit, P. Casal, M. Slemrod, H. Gouin, L. Truskinovsky, S. Benzoni-Gavage, D. Bresch, ...)

$$\tilde{e}\left(\rho, \nabla\rho\right)\left(= e\left(\rho, \nabla\rho\right)\right).$$

Bubbly fluid



Euler-type system (Iordansky, Kogarko, van Wijngaarden) :

$$\rho_t + \mathrm{div}(\rho \boldsymbol{v}) = 0, \quad (\alpha_g \rho_g)_t + \mathrm{div}(\alpha_g \rho_g \boldsymbol{v}) = 0,$$

$$N_t + \mathrm{div}(N\boldsymbol{v}) = 0, \quad (\rho \boldsymbol{v})_t + \mathrm{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v} + p\boldsymbol{I}) = 0,$$

$$p = p_g(a) + \rho_l \left(a\ddot{a} + \frac{3}{2}\dot{a}^2 \right), \quad \dot{a} = \frac{\partial a}{\partial t} + \boldsymbol{v} \cdot \nabla a,$$

$$\rho = \rho_l(1 - \alpha_g), \quad \alpha_g = \frac{4}{3}\pi a^3 N, \quad p_g(a) = p_0 \left(\frac{a_0}{a} \right)^{3\gamma}, \quad \gamma > 1.$$

Lagrangian for bubbly fluid equations, SG and V. Teshukov, 2001, CMT

$$\mathcal{L} = \int_{D_t} L \, dD,$$

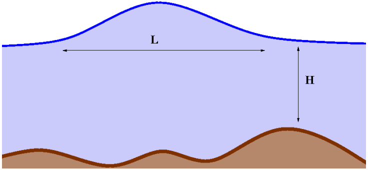
$$L(\boldsymbol{v}, a, \dot{a}, N) = \frac{\rho|\boldsymbol{v}|^2}{2} + 2\pi a^3 N \rho_l \dot{a}^2 - \alpha_g \rho_g \varepsilon_g(a).$$

Constraints :

$$\rho_t + \mathrm{div}(\rho \boldsymbol{v}) = 0, \quad N_t + \mathrm{div}(N\boldsymbol{v}) = 0, \quad (\alpha_g \rho_g)_t + \mathrm{div}(\alpha_g \rho_g \boldsymbol{v}) = 0.$$

Serre-Green-Naghdi equations

S. Busto, C. Escalante, M. Dumbser, N. Favrie and SG (JSC 2021)



$$\mathcal{L} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} L\, dx_1 dx_2,$$

$$L(\overline{\boldsymbol{v}}, h, \dot{h}, b, \dot{b}) = h\left(\frac{|\overline{\boldsymbol{v}}|^2}{2} + \frac{1}{6}\left(\dot{h} + \frac{3}{2}\dot{b}\right)^2 + \frac{1}{8}\dot{b}^2\right) - \frac{gh}{2}(h+2b) - Ch.$$

$\overline{\boldsymbol{v}}$-depth averaged velocity, h -fluid depth, $z = b(t, x_1, x_2)$ -bottom topography.

Constraints :

$$h_t + \mathrm{div}(h\overline{\boldsymbol{v}}) = 0.$$

1D Serre-Su-Gardner-Green-Naghdi equations

Equations in Eulerian coordinates

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = 0,$$

$$\frac{\partial hu}{\partial t} + \frac{\partial hu^2 + p}{\partial x} = 0.$$

h- fluid depth
$\overline{\boldsymbol{v}} = (u, 0)^T$,
u - average velocity

$$p = \frac{gh^2}{2} + \frac{1}{3}h^2\ddot{h}$$

$$\dot{h} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x},$$

$$\ddot{h} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\dot{h}.$$

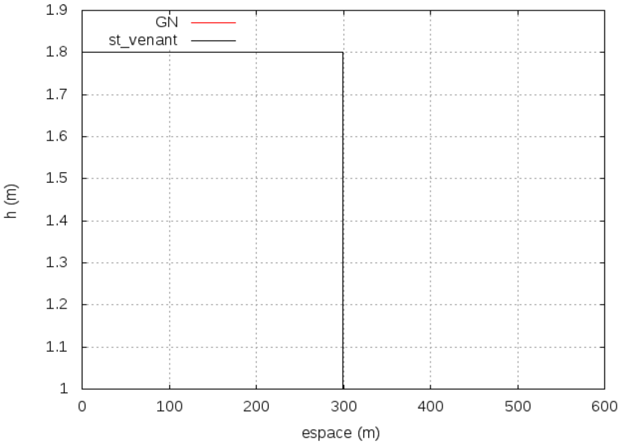
Lagrangian

$$\mathcal{L} = \int_{-\infty}^{\infty} \left(\frac{hu^2}{2} - W(h, \dot{h}) \right) dx,$$

$$W(h, \dot{h}) = \frac{gh^2}{2} - \frac{h\dot{h}^2}{6} = h\,\tilde{e}(h, \dot{h}).$$

$$p = h\frac{\delta W}{\delta h} - W.$$

Comparison of hyperbolic and dispersive waves



Dispersive shocks (A. V. Gurevich and L. Pitaevsky, A. L. Krylov, G. El, M. Hoefer, A. Kamchatnov, I. Baholdin, A. Chugainova, A. G. Kulikovskii, ...)

Mass Lagrangian coordinates

$$q = \int_0^X h_0(s)ds,$$

$$\mathcal{L} = \int_{-\infty}^{\infty} \left(\frac{u^2}{2} - \tilde{e}(\tau, \tau_t) \right) dq, \ \tilde{e} = \frac{W}{h},$$

$$u = x_t, \quad \frac{1}{h} = \tau = x_q, \quad \tilde{e}(\tau, \tau_t) = \frac{g}{2\tau} - \frac{1}{6} \left(\frac{\partial 1/\tau}{\partial t} \right)^2.$$

The governing equations are :

$$\tau_t - u_q = 0, \ u_t + p_q = 0,$$

with

$$p = -\frac{\delta \tilde{e}}{\delta \tau} = -\left(\frac{\partial \tilde{e}}{\partial \tau} - \frac{\partial}{\partial t} \left(\frac{\partial \tilde{e}}{\partial \tau_t} \right) \right) = \frac{g}{2\tau^2} + \frac{2}{3} \frac{\tau_t^2}{\tau^5} - \frac{1}{3} \frac{\tau_{tt}}{\tau^4}.$$

Energy conservation law

Then the general system admits the energy conservation law :

$$\left(e+\frac{1}{2}u^2\right)_t+(pu)_q=0,\ \ e=\tilde{e}-\frac{\partial\tilde{e}}{\partial\tau_t}\tau_t.$$

Inversion of an elliptic operator (O. Le Metayer, SG & S. Hank (2010))

System to solve :

$$\tau_t - u_q = 0,\ u_t - \left(\frac{\partial \tilde{e}}{\partial \tau} - \frac{\partial}{\partial t}\left(\frac{\partial \tilde{e}}{\partial \tau_t}\right)\right)_q = 0.$$

Or

$$\tau_t - u_q = 0,\ K_t - \left(\frac{\partial \tilde{e}}{\partial \tau}\right)_q = 0,$$

$$K = u + \left(\frac{\partial \tilde{e}}{\partial \tau_t}\right)_q = u - \frac{1}{3}\left(\frac{u_q}{\tau^4}\right)_q = \mathcal{A}u.$$

To find u we need to invert $\mathcal{A}$.

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- To use explicit methods to avoid expensive computations (inversion of elliptic operators)
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- To have a possibility to work with discontinuous initial data for dispersive equations (the famous Gurevich-Pitaevsky problem (Riemann problem) for the dispersive equations)

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- To be able to formulate "transparent" boundary conditions for non-linear dispersive equations (see such a difficulty for linear dispersive equations, C. Bethe, P. Noble, M. Kazakova, ...)
- To have a possibility to work with discontinuous initial data for dispersive equations (the famous Gurevich-Pitaevsky problem (Riemann problem) for the dispersive equations)
- One can use the whole arsenal of finite volume methods for hyperbolic equations and hence to share the pleasure of working on a new mathematical model with my 'hyperbolic' friends

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- The ‘ideal’ equations coming from physics have a variational formulation. So, it is natural to conserve such a property for an approximate system.
- We are sure to conserve basic properties of the approximate equations (in particular, the conservation of momentum and energy) due to the Noether theorem.
- The dissipation can easily be added in the Euler-Lagrange equations by using the Rayleigh functional approach.

Augmented lagrangian (penalty method) for non-dissipative dispersive equations

- At least a one-parameter family of ‘augmented’ lagrangians (containing new variables that is close to the ‘master’ lagrangian in some limit (for example, when the parameter goes to infnity).
- The Euler-Lagrange equations for the ‘augmented’ lagrangian should be unconditionally hyperbolic.

Application to SGN equations : 'master' lagrangian

$$q = \int_0^X h_0(s)ds,$$

$$\mathcal{L} = \int_{-\infty}^{\infty} \left(\frac{u^2}{2} - \tilde{e}(\tau, \tau_t) \right) dq,$$

$$u = x_t, \quad \frac{1}{h} = \tau = x_q, \quad \tilde{e}(\tau, \tau_t) = \frac{g}{2\tau} - \frac{h_t^2}{6}.$$

Governing equations :

$$\tau_t - u_q = 0, \quad u_t + p_q = 0,$$

with

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Augmented Lagrangian

$$\hat{\mathcal{L}} = \int_{-\infty}^{\infty} \left(\frac{x_t^2}{2} + \frac{\eta_t^2}{6} - \frac{g}{2\tau} - \lambda \frac{(\eta\tau - 1)^2}{6} \right) \, dq$$

Governing equations :

$$\begin{cases} \tau_t - u_q = 0, \\ u_t - \left(\frac{g}{\tau^3} + \frac{\lambda}{3}\eta^2 \right) \tau_q - \frac{\lambda}{3}(2\tau\eta - 1)\eta_q = 0, \\ \eta_{tt} = -\lambda \left(\eta\tau - 1 \right) \tau. \end{cases}$$

Characteristics :

$$\xi_{1,2} = 0, \quad \xi_{3,4} = \pm\sqrt{\frac{g}{\tau^3} + \frac{\lambda}{3}\eta^2}$$

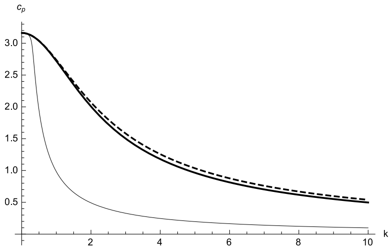
Phase velocity : $c_p = \frac{\omega}{k}$:

$$\left(c_p^{\pm}\right)^2 = \frac{\frac{g}{\tau_0^3} + \frac{\lambda}{3\tau_0^2} + \frac{\lambda\tau_0^2}{k^2} \pm \sqrt{\left(\frac{g}{\tau_0^3} + \frac{\lambda}{3\tau_0^2} + \frac{\lambda\tau_0^2}{k^2} \right)^2 - 4\frac{g\lambda}{\tau_0 k^2}}}{2}.$$

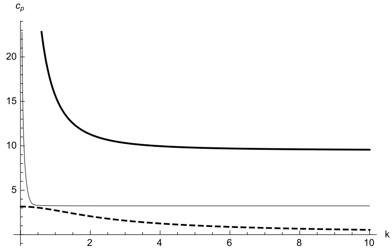
Dispersive properties

Phase velocity for the ‘master’ Lagrangian :

$$c_p^2 = \frac{g}{\tau_0^3 + \frac{k^2}{3\tau_0}}$$



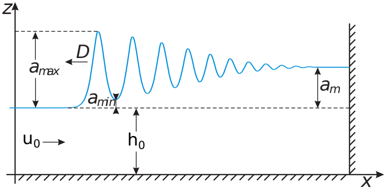
$(c_p^-)^2$

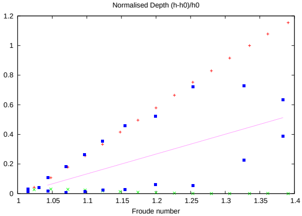


$(c_p^+)^2$

$\tau_0 = 1\ m^{-1}$, $\lambda = 1\ m^2\ s^{-2}$, $\lambda = 160\ m^2\ s^{-2}$

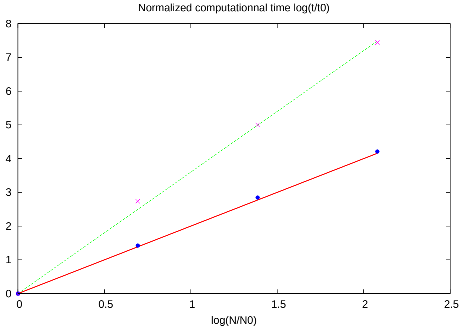
Favre waves (N. Favrie and SG, Nonlinearity, 2017)





Comparison between the experimental results (squares) and numerical results. The agreement is perfect until the Froude number about 1.25. After this critical value, the model is no more valid, the wave breaking occurs which is not described by the SGN model (see SG, Liapidevskii and Chesnokov, 2016, 2017 for a two-layer model). The middle straight line corresponds to the solution of the Saint-Venant equations.

Favre waves



The computational time : $T \approx N^{3.6}$ (inversion) and $T \approx N^2$ (hyperbolic).

Numerical results 2D

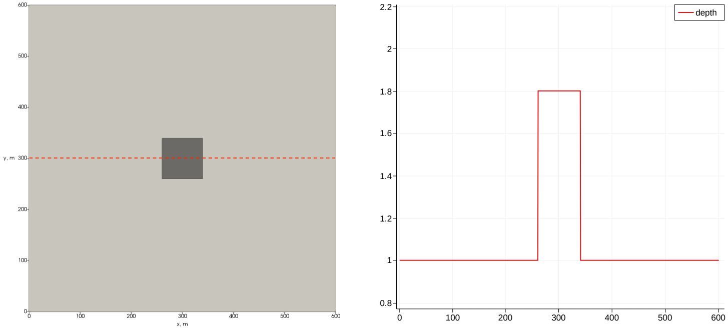


Figure – Initial condition of a 2D Riemann problem for the extended SGN model. Initial water depth is $h_l = 1.8\ m$ inside the dark gray region $\{(x, y) : 260\ m < x, y < 340\ m\}$ and $h_r = 1.0\ m$ outside. Initial velocity is $u_0 = v_0 = 0\ m/s$.

Numerical results 2D

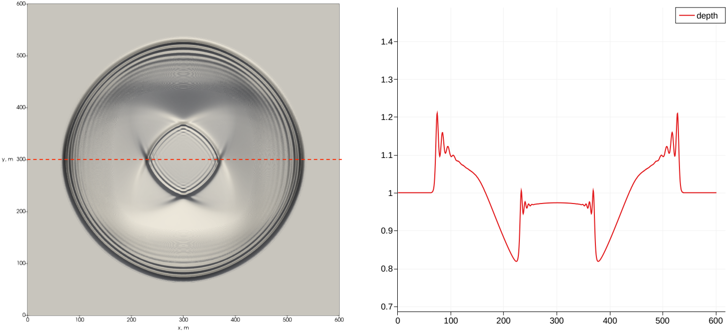


Figure – Numerical solution to 2D Riemann problem for the extended SGN model, $t = 20s$.

Solitary wave interaction with an island

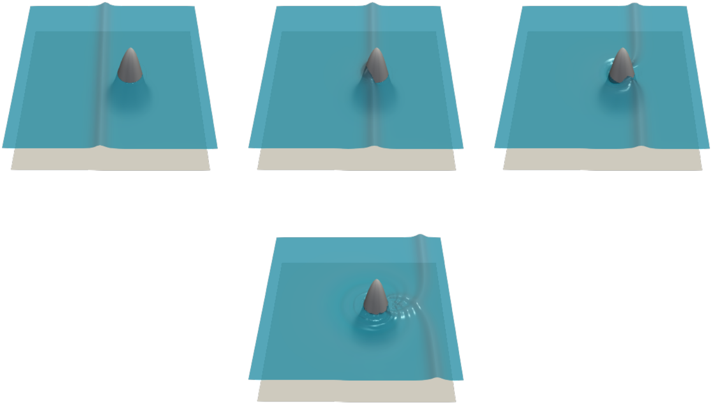


Figure – Interaction of a solitary wave with an island (S. Busto, C. Escalante, M. Dumbser, N. Favrie and SG (JSC 2021)).

Rigorous justification of the extended Lagrangian method for the Serre-Green-Naghdi equations

V. Duchene, Nonlinearity, 2019

Space dispersion : the defocusing cubic NLS equation

$$\boxed{i\psi_t + \frac{1}{2}\Delta\psi - |\psi|^2\,\psi = 0}$$

- A wide range of applications : quantum fluids, nonlinear optics, surface gravity waves
- Advantage : the equation is integrable in 1D case [Zakharov, Manakov 1974]

The Madelung transform

$$\psi(\boldsymbol{x}, t) = \sqrt{\rho(\boldsymbol{x}, t)}e^{i\theta(\boldsymbol{x},t)} \qquad \boldsymbol{u} = \nabla\theta$$

$$\begin{cases} \rho_t + \mathrm{div}(\rho\boldsymbol{u}) = 0 \\ (\rho\boldsymbol{u})_t + \mathrm{div}\left(\rho\boldsymbol{u}\otimes\boldsymbol{u} + \Pi\right) = 0 \end{cases}$$

$$\text{with :} \quad \Pi = \left(\frac{\rho^2}{2} - \frac{1}{4}\Delta\rho\right)\boldsymbol{I} + \frac{1}{4\rho}\nabla\rho\otimes\nabla\rho$$

Lagrangian for NLS equation

For the previous set of equations, we can construct the Lagrangian :

$$\mathcal{L}(\boldsymbol{u}, \rho, \nabla\rho) = \int_{\Omega_t} \left(\rho \frac{|\boldsymbol{u}|^2}{2} - \frac{\rho^2}{2} - \frac{1}{4\rho} \frac{|\nabla\rho|^2}{2} \right) d\Omega_t$$

Energy conservation law :

$$\frac{\partial E}{\partial t} + \operatorname{div}\left(E\boldsymbol{u} + \Pi\boldsymbol{u} - \frac{1}{4}\dot{\rho}\nabla\rho\right) = 0, \quad \dot{\rho} = \rho_t + \boldsymbol{u}\cdot\nabla\rho$$

where

$$E = \rho \frac{|\boldsymbol{u}|^2}{2} + \frac{\rho^2}{2} + \frac{1}{4\rho} \frac{|\nabla\rho|^2}{2}$$

'Augmented' Lagrangian

F. Dhaouadi, N. Favrie, SG (SAM, 2019)

$$\hat{L} = \int_{\Omega_t} \left(\rho \frac{|\boldsymbol{u}|^2}{2} - \frac{\rho^2}{2} - \frac{1}{4\rho} \frac{|\boldsymbol{p}|^2}{2} - \frac{\lambda}{2} \rho \left(\frac{\eta}{\rho} - 1 \right)^2 + \frac{\beta \rho}{2} w^2 \right) d\Omega_t$$

$$\boldsymbol{p} = \nabla \eta \qquad w = \dot{\eta}$$

$\frac{\lambda}{2} \rho \left(\frac{\eta}{\rho} - 1 \right)^2$: Penalty $\qquad$ $\frac{\beta \rho}{2} \dot{\eta}^2$: Regularizer

Weakly hyperbolic Galilean invariant augmented system

$$
\begin{cases}
\rho_t + \mathrm{div}(\rho \boldsymbol{u}) = 0 \\
(\rho \boldsymbol{u})_t + \mathrm{div}\left(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \Pi\right) = 0 \\
(\rho \eta)_t + \mathrm{div}(\rho \eta \boldsymbol{u}) = \rho w \\
(\rho w)_t + \mathrm{div}\left(\rho w \boldsymbol{u} - \dfrac{1}{4\rho\beta}\boldsymbol{p}\right) = \dfrac{\lambda}{\beta}\left(1 - \dfrac{\eta}{\rho}\right) \\
\dfrac{\partial \boldsymbol{p}}{\partial t} + \boldsymbol{\nabla}\left(\boldsymbol{p}\cdot\boldsymbol{u} - w\right) + \left(\dfrac{\partial \boldsymbol{p}}{\partial \boldsymbol{x}} - \left(\dfrac{\partial \boldsymbol{p}}{\partial \boldsymbol{x}}\right)^T\right)\boldsymbol{u} = 0.
\end{cases}
$$

$$
(\rho E)_t + \boldsymbol{\nabla}\cdot\left(\rho \boldsymbol{u} E + \Pi \boldsymbol{u} - \frac{1}{4\rho} w \boldsymbol{p}\right) = 0,
$$

$$
E = \frac{|\boldsymbol{u}|^2}{2} + \frac{\beta}{2}\dot{\eta}^2 + \frac{\rho}{2} + \frac{1}{4\rho^2}\frac{|\boldsymbol{p}|^2}{2} + \frac{\lambda}{2}\left(\frac{\eta}{\rho} - 1\right)^2
$$

$$
\Pi = \left(\frac{\rho^2}{2} - \frac{1}{4\rho}|\boldsymbol{p}|^2 + \eta\lambda\left(1 - \frac{\eta}{\rho}\right)\right)\boldsymbol{I} + \frac{1}{4\rho}\boldsymbol{p}\otimes\boldsymbol{p}
$$

Clearning procedure (hyperbolic system)

$$\begin{cases} \rho_t + \mathrm{div}(\rho \boldsymbol{u}) = 0 \\ (\rho \boldsymbol{u})_t + \mathrm{div}\left(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \Pi\right) = 0 \\ (\rho\eta)_t + \mathrm{div}(\rho\eta \boldsymbol{u}) = \rho w \\ (\rho w)_t + \mathrm{div}\left(\rho w \boldsymbol{u} - \dfrac{1}{4\rho\beta}\boldsymbol{p}\right) = \dfrac{\lambda}{\beta}\left(1 - \dfrac{\eta}{\rho}\right) \\ \dot{\boldsymbol{p}} + \left(\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T \boldsymbol{p} - \nabla w + 2a_c\, \rho\, \mathrm{curl}(\boldsymbol{\psi}) = 0 \\ \rho\dot{\boldsymbol{\psi}} - \dfrac{a_c}{2}\mathrm{curl}(\boldsymbol{p}) = 0. \end{cases}$$

$$\Pi = \left(\frac{\rho^2}{2} - \frac{1}{4\rho}|\boldsymbol{p}|^2 + \eta\lambda\left(1 - \frac{\eta}{\rho}\right)\right)\boldsymbol{I} + \frac{1}{4\rho}\boldsymbol{p} \otimes \boldsymbol{p}$$

Energy equation

Energy equation :

$$\left(\rho(E + \|\boldsymbol{\psi}\|^2/2)\right)_t + \nabla \cdot \left(\rho \boldsymbol{u}(E + \|\boldsymbol{\psi}\|^2/2) + \Pi \boldsymbol{u} - \frac{1}{4\rho} w \boldsymbol{p} + \frac{a_c}{2} \boldsymbol{\psi} \times \boldsymbol{p}\right) = 0,$$

$$E = \frac{|\boldsymbol{u}|^2}{2} + \frac{\beta}{2}\dot{\eta}^2 + \frac{\rho}{2} + \frac{1}{4\rho^2}\frac{|\boldsymbol{p}|^2}{2} + \frac{\lambda}{2}\left(\frac{\eta}{\rho} - 1\right)^2$$

Clearning procedure

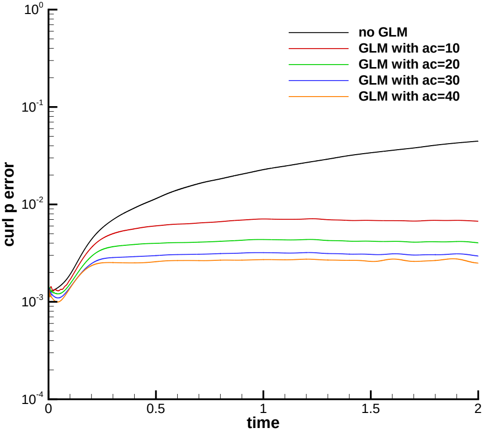


Figure – L_2 norm of $curl(\boldsymbol{p})$ as a function of the cleaning speed a_c. An exact solution was tested (S. Busto, C. Escalante, M. Dumbser, N. Favrie, SG *et al.* 2021)

DSW Numerical results

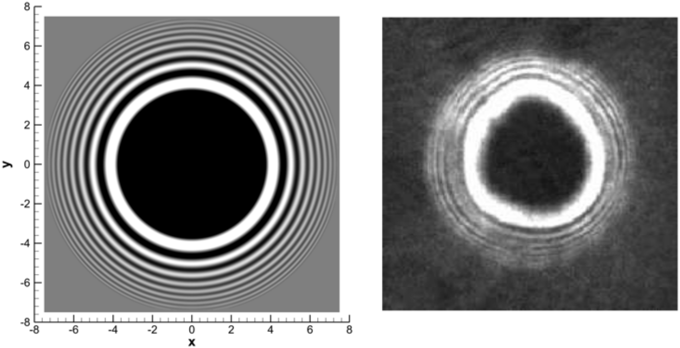


Figure – Blast waves in the Bose-Einstein condensates (S. Busto *et al.* 2021 – numerics), M. A. Hoefer, M. J. Ablowitz, I. Coddington, E. A. Cornell, P. Engels, and V. Schweikhard (2006, experiment)

Strange question

Are there other reasons why we need a hyperbolic regularization ?

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Are there other reasons why we need a hyperbolic regularization ?

Yes ! One can have discontinuous solutions of the dispersive equations and this is why we need a hyperbolic approximation !

Toy model : Benjamin-Bona-Mahony (BBM) equation

$$u_t + \left(\frac{u^2}{2} - \varepsilon^2 u_{tx}\right)_x = 0.$$

BBM (P. Olver, 1980)

$$\mathcal{L} = -\frac{\varphi_t \varphi_x}{2} - \frac{\varphi_x^3}{6} + \varepsilon^2 \frac{\varphi_t \varphi_{xxx}}{2}, \quad u = \varphi_x.$$

$$u_t = -\left(1 - \varepsilon^2 \frac{\partial^2}{\partial x^2}\right)^{-1} \frac{\partial}{\partial x}\left(\frac{\delta H}{\delta u}\right), \quad H = \int_{-\infty}^{\infty} \frac{u^3}{6}\, dx.$$

Conservation laws for the BBM equation (Olver, 1979)

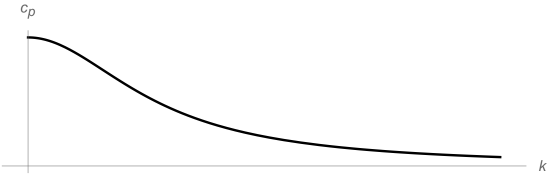
$$(u - u_{xx})_t + \left(\frac{u^2}{2}\right)_x = 0,$$

$$\left(\frac{u^2}{2} + \frac{u_x^2}{2}\right)_t + \left(\frac{u^3}{3} - u u_{tx}\right)_x = 0,$$

$$\left(\frac{u^3}{3}\right)_t - \left(u_t^2 - u_{xt}^2 + u^2 u_{xt} - \frac{u^4}{4}\right)_x = 0.$$

Dispersion relation (linearization near the constant state $u = c_0$) :

$$c_p = \frac{c_0}{1 + \varepsilon^2 k^2}$$

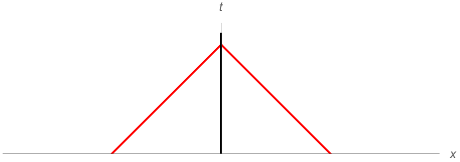


The phase and group velocities are bounded : $0 < c_p < c_0$!

Discontinuous solutions of the BBM equation

$$u_t + \left(\frac{u^2}{2} - \varepsilon^2 u_{tx}\right)_x = 0.$$

Stationary discontinuous weak solutions :



What about more general solutions ? For example, a shock connecting constant states and highly oscillating zones ? What are the Rankine - Hugoniot conditions on such a shock ?

'Broken' extremal conditions

$$a[u] = \int \mathcal{L}(u, u')dx,$$

$$\delta a = \int \left(\frac{\delta \mathcal{L}}{\delta u} \delta u + \frac{d}{dx} \left(\frac{\partial L}{\partial u'} \delta u \right) \right) dx, \quad \frac{\delta \mathcal{L}}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial u'} \right).$$

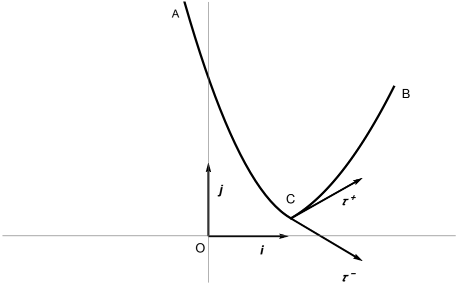
In the case of non-smooth ("broken") extremal curves, the same Euler - Lagrange equation should be satisfied for each smooth part of the extremal curve :

$$\frac{\delta \mathcal{L}}{\delta u} = 0. \tag{3}$$

Together with (3) an additional condition should also be satisfied at the "broken" point :

$$\left[\frac{\partial \mathcal{L}}{\partial u'} \right] = 0. \tag{4}$$

GRH conditions : a mass sliding over a heavy string



$$W[y] = g\gamma \int_A^B y(s)ds + mgy(C), \quad \int_A^B ds = l.$$

Piecewise smooth curve (catenary).
GRH condition (Weierstrass-Erdmann condition, corner condition, broken extremal condition) :

$$\boldsymbol{\tau}^+ \cdot i = \boldsymbol{\tau}^- \cdot i$$

Travelling waves (with $\varepsilon = 1$)

Let $\xi = x - Dt$, $' = \dfrac{d}{d\xi}$

Periodic solutions

$$(u_1 + u_2 + u_3)u'^2 = P(u) \equiv (u - u_1)(u - u_2)(u_3 - u),\ 0 < u_1 \leq u_2 \leq u_3.$$

$$a[u] = \int \mathcal{L}(u, u')dx, \quad \mathcal{L}(u, u') = \frac{Du'^2}{2} + \frac{P(u)}{6}, \quad D = \frac{1}{3}(u_1 + u_2 + u_3).$$

Averaging :

$$\overline{f(u)} = \frac{1}{L}\int_0^L f(u(\xi))d\xi.$$

GRH conditions for the BBM equation : travelling waves

$$-D[u] + [u^2/2 + Du''] = 0, \quad [u'] = 0, \quad ' = \frac{d}{d\xi}, \quad \xi = x - Dt.$$

Here $[f] = f^+ - f^-$.

GRH = fulfillment of conservation law + extra condition. If we have a shock connecting a constant state with a non-trivial solution, the extra condition is simply $u' = 0$.

GRH conditions for the BBM equation : travelling waves

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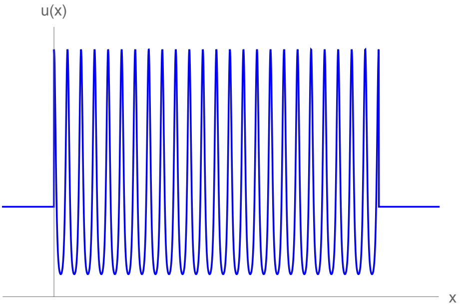
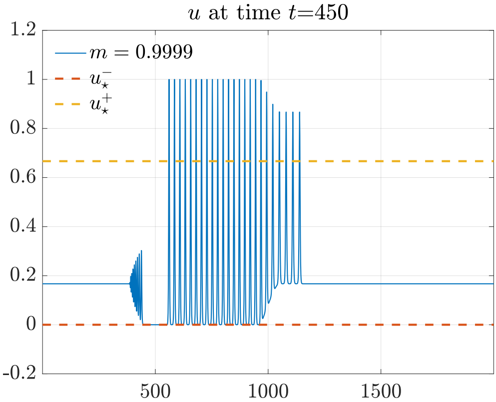
GRH = fulfillment of conservation law + extra condition. If we have a shock connecting a constant state with a non-trivial solution, the extra condition is simply $u'=0$.

Example : Initial data

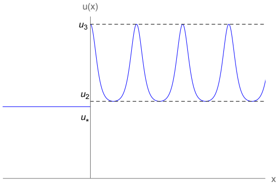


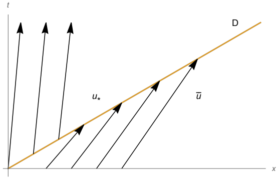
Example : solution



Weak solution

SG and K.-M. Shyue, Nonlinearity (2022)





$$\text{Stability condition}: \overline{u} < D < u_\star.$$

Extra RH condition for general dispersive systems

Extra relation for bubbly fluids :

$$[\dot{\rho}] = 0. \tag{5}$$

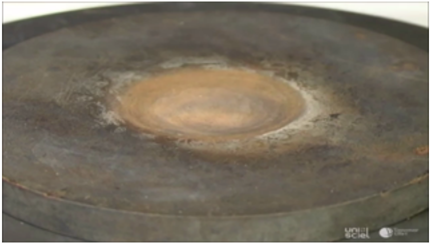
Extra relation for capillary fluids

$$\left[\frac{d\rho}{dn}\right] = 0, \quad \text{or} \quad \left[\frac{d\rho}{dx}\right] = 0 \quad \text{in 1D case} \tag{6}$$

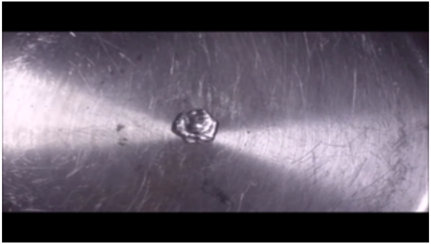
Applications : Leidenfrost phenomenon (SG, H. Gouin, Phys. Rev. E, 2022)

Johann Gottlob Leidenfrost (1715 – 1794) was born in Rosperwenda (now Berga am Elster in Thuringia, Germany)

Leidenfrost's droplets



Leidenfrost's stars



Leidenfrost's puddles

Mathematical model : 1D Euler-Korteweg + van der Waals equation of state + heat transfer effects.

$$W(\rho, \nabla\rho) = \rho\,\alpha(\rho,\eta) + \frac{\lambda}{2}\rho_x^2, \quad \lambda = const. \tag{7}$$

The GRH relation :

$$[\rho_x] = 0 \tag{8}$$

This condition allows us to find Leidenfrost temperature about 180 ^oC and the temperature inside of the water about 94 ^oC. It corresponds to experimental values !

Further development of the hyperbolic approximation

Sometimes, the variational formulation is absent.

From D. R. Scott, D.J. Stevenson and J. A. Whitehead, 1986, Nature.

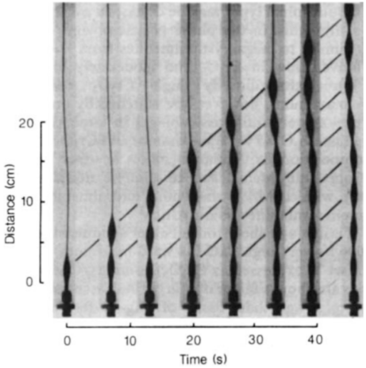


Fig. 2 The formation of a periodic wavetrain. This experiment started with a uniform pipe, sustained by a constant supply of buoyant liquid from the nozzle seen at the base of the frames. The rate of supply was then increased and held at a new constant level, approximately 220 times higher. This increase could be accommodated by the formation of a larger uniform pipe, but the system preferentially forms a periodic wavetrain. In longer experiments, the wavetrain does slowly degrade down to a larger uniform pipe. The ruled lines are parallel and equally spaced.

Conduit equation

The conduit equation is written for the cross-sectional area $u > 0$:

$$u_t + (u^2 + u_x u_t - u u_{tx})_x = 0$$

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Maiden, M. D. *et al* (2016, 2018, 2020), Maiden M. D. and Hoefer, M. A. (2016), M. A. Johnson and W.R. Perkins (2020), ...

General properties of the conduit equation

- Only one additional conservation law exists (R. Khamitova, 2009 ; N. Mindu and D. P. Maison, 2014)

$$\left(\frac{1}{u}+\frac{u_{xx}}{u}\right)_t-\left(2\ln\left(u\right)\right)_x=0.$$

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Periodic and solitary wave solutions

$$Du'^2 = P(u) = Cu^2 - 2u^2\ \ln(u) - 2Du - Q, \quad ' = \frac{d}{d\xi}, \quad \xi = x - Dt.$$

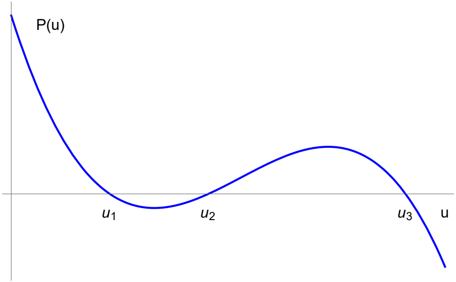


Figure – A typical behavior of the function $P(u)$ is shown. In a domain of parameters $C, D > 0, Q < 0$ it has three roots $0 < u_1 < u_2 < u_3$. The periodic solution oscillates between u_2 and u_3.

Solitary wave solutions

For solitary waves ($u_1 = u_2 < u_3$), the wave velocity D is linked to amplitude $a = u_3 - u_2$ by an explicit relation (Olson and Christensen, 1986) and satisfies the inequality :

$$D > 2\, u_2 = 2\, \overline{u}.$$

For small amplitudes, one has a classical relation :

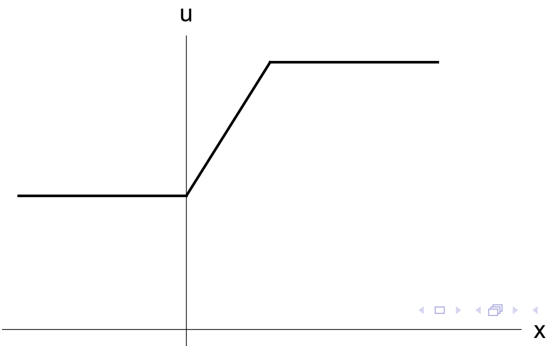
$$D = 2\left(u_2 + \frac{a}{3}\right) + \mathcal{O}(a^2).$$

Singular solutions : rarefaction fans

$$u_t + (u^2 + u_x u_t - u u_{tx})_x = 0$$

Rarefaction fans

$$u(t,x) = \begin{cases} 1/2, & x/t < 1 \\ x/(2t) & 1 < x/t < 2 \\ 1 & x/t > 2 \end{cases}$$



Singular traveling wave solutions

One can link a smooth periodic solution to a constant state by the Rankine-Hugoniot conditions (cf. the BBM equation and the Serre-Green-Naghdi equations describing surface gravity waves, SG *et al.* 2020, SG and K. - M. Shyue, 2021).

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$$u_t + (u^2 + u_t u_x - u u_{tx})_x = 0$$

$$-D[u] + [u^2 - Du'^2 + Duu''] = 0, \quad ' = \frac{d}{d\xi}, \quad \xi = x - Dt.$$

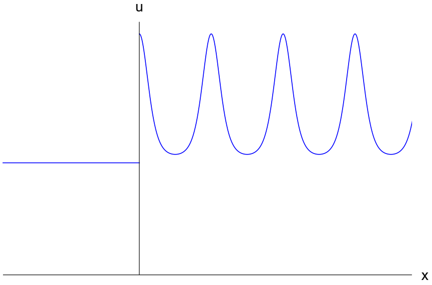
Here $[f] = f^+ - f^-$.

Singular traveling wave solutions

Can we link each periodic solution of the conduit equation (so, for given u_1, u_2 and u_3) to a constant state ?
Experimental facts :

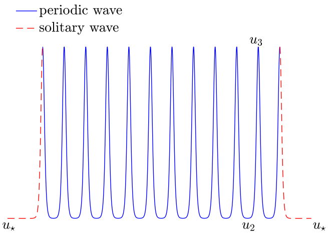
- $D_s = D_p = D$.
- The periodic waves should be quite long ($D_p > 2\overline{u}$).
- "Corner condition" : $[u'] = 0$.
- There are two possible constant states satisfying the Rankine-Hugoniot relations. Only one of them can be linked 'in a stable way'.

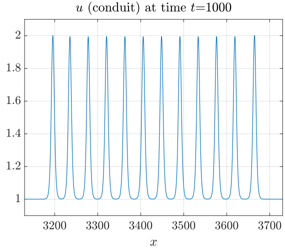
Singular traveling wave solutions



Stable multi-hump solitary waves (Frankenstein's monsters)

The generalized RH conditions allow us to construct multi-hump solitary wave solutions of dispersive equations having the state $u_\star$ at infinity : $u_1 < u_\star < u_2 < u_3$.





Numerical calculations are tricky

Hyperbolic–elliptic splitting (O. Lemetayer *et al.* (2010) for the SGN equations, SG and K.- M. Shyue (2022) for the BBM equation)

$$K_t - (2\ln(u))_x = 0, \quad K = \frac{1}{u} + \frac{u_{xx}}{u}.$$

Another idea ?

Can the conduit equation be approximated by a system of hyperbolic equations ?

Another idea ?

Can the conduit equation be approximated by a system of hyperbolic equations ?
A reversible equation should be replaced by a reversible hyperbolic system...

Godunov's systems

In 1961 S. K. Godunov proposed the following abstract form of a system of conservation laws for the vector variable $\boldsymbol{v} = (v_1, \ldots, v_n)^T$:

$$\frac{\partial}{\partial x^\beta}\left(\frac{\partial L^\beta}{\partial v^\alpha}\right) = 0, \tag{9}$$

with $\beta = 0, \ldots, m$, $\alpha = 1, \ldots, n$ (x^0 corresponds to the time derivative). Here L^β are functions of $\boldsymbol{v}$, and the summation is taken over repeated indexes. It admits an additional conservation law :

$$\frac{\partial E^\beta}{\partial x^\beta} = 0, \quad E^\beta = v^\alpha \frac{\partial L^\beta}{\partial v^\alpha} - L^\beta, \tag{10}$$

If E^0 is convex, the equations are t - *hyperbolic* in the sense of Friedrichs. A number of *reversible* models of continuum mechanics can be written in Godunov's form (Godunov and Romenskii, 2003, Peshkov and Romenskii, ...).

Generalization of Godunov's systems (SG, S. Shugrin, 1996)

$$\frac{\partial}{\partial x^{\beta}}\left(\frac{\delta L^{\beta}}{\delta v^{\alpha}}\right)=0. \tag{11}$$

Here we used usual notation for the variational derivative :

$$\frac{\delta L^{\beta}}{\delta v^{\alpha}}=\frac{\partial L^{\beta}}{\partial v^{\alpha}}-\frac{\partial}{\partial x^{\gamma}}\left(\frac{\partial L^{\beta}}{\partial v^{\alpha}_{,\gamma}}\right).$$

Equations (11) also admit an additional conservation law

$$\frac{\partial E^{\beta}}{\partial x^{\beta}}=0, \quad E^{\beta}=v^{\alpha}\frac{\delta L^{\beta}}{\partial v^{\alpha}}-L^{\beta}+v^{\alpha}_{,\gamma}\frac{\partial L^{\beta}}{\partial v^{\alpha}_{,\gamma}}. \tag{12}$$

Conduit equation in Godunov's form

- We introduce a new variable v instead of u : $u = \sqrt{1+2v}$, $t = x^0$, $x = x^1$.
- Let

$$L^0(v, v_x) = L(v, v_x) = \sqrt{1+2v} - \frac{v_x^2}{2(1+2v)}, \tag{13}$$

$$L^1(v) = M(v) = -\frac{1}{2}(1+2v)(\ln(1+2v)-1).$$

- Then

$$\left(\frac{\delta L}{\delta v}\right)_t + \left(\frac{\partial M}{\partial v}\right)_x = 0$$

‘Hyperbolized’ conduit equation in Godunov’s form

Consider a two-parameter family of potentials

$$\mathcal{L}(v, z, z_x, z_t) = \sqrt{1+2v} + \frac{z_t^2}{2c^2} - \frac{z_x^2}{2} - \frac{\lambda}{2}\left(z - \sqrt{1+2v}\right)^2, \qquad (14)$$

where λ and c are large parameters. Let us replace the conduit equation in v by a system of equations for two unknowns v and z :

$$\left(\frac{\partial \mathcal{L}}{\partial v}\right)_t + \left(\frac{\partial M}{\partial v}\right)_x = 0, \quad \frac{\delta \mathcal{L}}{\delta z} = 0, \qquad (15)$$

with

$$\frac{\delta \mathcal{L}}{\delta z} = \mathcal{L}_z - (\mathcal{L}_{z_t})_t - (\mathcal{L}_{z_x})_x.$$

It admits the conservation law :

$$\left(z_t\mathcal{L}_{z_t} + v\mathcal{L}_v - \mathcal{L}\right)_t + \left(z_t\mathcal{L}_{z_x} + vM_v - M\right)_x = 0. \qquad (16)$$

'Hyperbolized' conduit equation in Godunov's form

The first equation of (15) becomes :

$$\left(\frac{1}{\sqrt{1+2v}} + \lambda\frac{z-\sqrt{1+2v}}{\sqrt{1+2v}}\right)_t - \frac{2}{1+2v}v_x = 0. \tag{17}$$

The second equation of (15) becomes :

$$-\frac{1}{c^2}z_{tt} + z_{xx} = \lambda(z-\sqrt{1+2v}). \tag{18}$$

Finally, we return back to u-variable ($u=\sqrt{1+2v}$) :

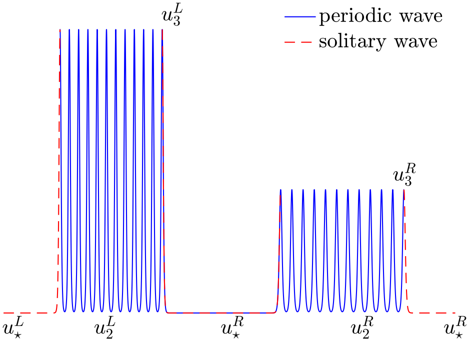
$$\left(\frac{1}{u} + \lambda\frac{z-u}{u}\right)_t - \frac{2u_x}{u} = 0, \quad -\frac{1}{c^2}z_{tt} + z_{xx} = \lambda(z-u). \tag{19}$$

'Hyperbolized' conduit equation in Godunov's form

The eigenvalues are $\pm c$ and $2u/(1+\lambda z)$. Hence, for large enough c and λ, the equations are hyperbolic. The initial conditions for the hyperbolic 'conduit' system are :

$$u(0,x) = u_0(x), \quad z(0,x) = u_0(x), \quad p(0,x) = \frac{du_0(x)}{dx}. \tag{20}$$

Interaction of Frankenstein's monsters



Interaction of Frankenstein's monsters

Multi-hump solitary waves interact as almost solitary waves.

Euler equations + heat conduction, Clausius-Duhem inequality

The Euler equations augmented by the Fourier heat conduction terms (applications : laser beam of high intensity interacting with a dense material, cf. O. Le Metayer, R. Saurel, JFM, 2006 ; Applications : laser beam of high intensity interacting with a dense material, cf. O. Le Metayer, R. Saurel, JFM, (2006) ; plasma physics, cf. Q. Wargnier, S. Faure, B. Graille, T. Magin and M. Massot, SIAM J. Sci. Comput. (2020) ; the BGK model, cf. W. Haegeman, G. Orlando, M. Massot, S. Kokh (2024))

Euler-Fourier equations

$$\rho_t + \mathrm{div}(\rho \boldsymbol{u}) = 0, \tag{21}$$

$$(\rho \boldsymbol{u})_t + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u} + p\boldsymbol{I}) = 0, \tag{22}$$

$$E_t + \mathrm{div}\left(E\boldsymbol{u} + p\boldsymbol{u} - K\nabla\theta\right) = 0, \tag{23}$$

$$E = \frac{\rho|\boldsymbol{u}|^2}{2} + \rho\varepsilon(v,\eta), \qquad \theta d\eta = d\varepsilon + p\, dv, \quad v = \frac{1}{\rho}. \tag{24}$$

Clausius – Duhem inequality

$$(\rho\eta)_t + \mathrm{div}(\rho\eta\boldsymbol{u} - \frac{K}{\theta}\nabla\theta) = \frac{K}{\theta^2}|\nabla\theta|^2 \geq 0$$

Modified Hamilton's principle for the Euler equations + the heat conduction

$$\mathcal{L} = \int_{D_t} \left(\frac{1}{2}\rho|\boldsymbol{u}|^2 - \frac{1}{2}\alpha(\rho)|\nabla\phi|^2 - \rho\varepsilon^\star(\rho, \dot{\phi}) \right) dD, \tag{25}$$

Constraint

$$\rho_t + \mathrm{div}(\rho\boldsymbol{u}) = 0.$$

Modified Hamilton's principle for the Euler equations + the heat conduction

$$\mathcal{L} = \int_{D_t} \left(\frac{1}{2}\rho|\boldsymbol{u}|^2 - \frac{1}{2}\alpha(\rho)|\nabla\phi|^2 - \rho\varepsilon^{\star}(\rho, \dot{\phi}) \right) dD, \tag{25}$$

Constraint

$$\rho_t + \mathrm{div}(\rho\boldsymbol{u}) = 0.$$

- Virtual displacements $\boldsymbol{\zeta}$
- Virtual thermal displacement $\delta\phi$

Governing equations for the modified Euler+ heat conduction equations

$$
\begin{gathered}
\rho_t + \operatorname{div}(\rho \boldsymbol{u}) = 0, \\
(\rho \boldsymbol{u})_t + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \Pi) = 0, \\
\Pi = \left(p + \frac{1}{2} \left(\rho \alpha'(\rho) - \alpha(\rho) \right) |\nabla \phi|^2 \right) \boldsymbol{I} + \alpha(\rho) \nabla \phi \otimes \nabla \phi, \\
(\rho \eta)_t + \operatorname{div}(\rho \eta \boldsymbol{u} + \alpha(\rho) \nabla \phi) = 0.
\end{gathered}
$$

Equation for the gradient of the thermal displacement

We denote

$$\boldsymbol{j} = \nabla \phi.$$

Taking the gradient of the equation

$$\dot{\phi} + \theta = 0$$

one gets

$$\boldsymbol{j}_t + \nabla \left(\boldsymbol{j} \cdot \boldsymbol{u} + \theta \right) = 0.$$

Reversible system with $\boldsymbol{j}$ notations

$$\rho_t + \mathrm{div}\,(\rho \boldsymbol{u}) = 0,$$

$$(\rho \boldsymbol{u})_t + \mathrm{div}\,(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \Pi) = 0,$$

$$\Pi = \left(p + \frac{1}{2}\left(\rho\alpha'(\rho) - \alpha(\rho)\right)|\boldsymbol{j}|^2\right)\boldsymbol{I} + \alpha(\rho)\boldsymbol{j} \otimes \boldsymbol{j},$$

$$(\rho\eta)_t + \mathrm{div}\,(\rho\eta\boldsymbol{u} + \alpha(\rho)\boldsymbol{j}) = 0,$$

$$\boldsymbol{j}_t + \nabla\left(\boldsymbol{j}\cdot\boldsymbol{u} + \theta\right) + \left(\frac{\partial \boldsymbol{j}}{\partial \boldsymbol{x}} - \left(\frac{\partial \boldsymbol{j}}{\partial \boldsymbol{x}}\right)^T\right)\boldsymbol{u} = 0, \quad \mathrm{curl}\boldsymbol{j} = 0.$$

Energy conservation :

$$E_t + (E\boldsymbol{u} + \Pi\boldsymbol{u} + \alpha(\rho)\theta\boldsymbol{j}) = 0, \quad E = \frac{1}{2}\rho|\boldsymbol{u}|^2 + \rho\varepsilon + \frac{1}{2}\alpha|\boldsymbol{j}|^2.$$

Irreversible system with $\boldsymbol{j}$ notations

A special attention should be payed to the case of dissipative equations, some basic thermodynamics should also be respected. For example, if a Lyapunov functional exists for the basic system, it should also exist for an extended system.

$$\rho_t + \operatorname{div}(\rho \boldsymbol{u}) = 0,$$

$$(\rho \boldsymbol{u})_t + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \Pi) = 0,$$

$$\Pi = \left(p + \frac{1}{2}\left(\rho\alpha'(\rho) - \alpha(\rho)\right)|\boldsymbol{j}|^2\right)\boldsymbol{I} + \alpha(\rho)\boldsymbol{j} \otimes \boldsymbol{j}$$

$$(\rho\eta)_t + \operatorname{div}(\rho\eta\boldsymbol{u} + \alpha(\rho)\boldsymbol{j}) = \frac{\alpha(\rho)}{\theta}\frac{|\boldsymbol{j}|^2}{\tau} \geq 0,$$

$$\boldsymbol{j}_t + \nabla(\boldsymbol{j}\cdot\boldsymbol{u} + \theta) + \left(\frac{\partial \boldsymbol{j}}{\partial \boldsymbol{x}} - \left(\frac{\partial \boldsymbol{j}}{\partial \boldsymbol{x}}\right)^T\right)\boldsymbol{u} = -\frac{\boldsymbol{j}}{\tau}.$$

Energy conservation :

$$E_t + (E\boldsymbol{u} + \Pi\boldsymbol{u} + \alpha(\rho)\theta\boldsymbol{j}) = 0, \quad E = \frac{1}{2}\rho|\boldsymbol{u}|^2 + \rho\varepsilon + \frac{1}{2}\alpha|\boldsymbol{j}|^2.$$

1D irreversible case

$$\rho_t + (\rho u)_x = 0,$$

$$(\rho u)_t + \left(\rho u^2 + \Pi\right)_x = 0, \quad \Pi = p + \frac{1}{2}\left(\rho\alpha'(\rho) + \alpha(\rho)\right) j^2,$$

$$(\rho\eta)_t + (\rho\eta u + \alpha(\rho) j)_x = \frac{\alpha(\rho)}{\theta}\frac{j^2}{\tau} \geq 0,$$

$$j_t + (ju + \theta)_x = -\frac{j}{\tau}.$$

Energy conservation :

$$E_t + \left(Eu + \Pi u + \alpha(\rho)\,\theta j\,\right)_x = 0, \quad E = \frac{1}{2}\rho u^2 + \rho\varepsilon + \frac{1}{2}\alpha(\rho) j^2.$$

Hyperbolicity condition

Let $\varepsilon_{vv} > 0$, $\varepsilon_{vv}\varepsilon_{\eta\eta} - \varepsilon_{v\eta}^2 > 0$, $v = 1/\rho$, and $\dfrac{d^2}{d\rho^2}\left(\dfrac{1}{\alpha(\rho)}\right) \geq 0$.

Then the equations are hyperbolic.

Eigenvalues :

$$\lambda_1 = u - c_a < \lambda_2 = u - c_t < \lambda_3 = u + c_t < \lambda_4 = u + c_a.$$

Hyperbolicity condition (particular case)

A particular case $\alpha(\rho) = \dfrac{\varkappa^2}{\rho}$, $\varkappa^2 > 0$.

$$c_{t,a}^2 = \frac{p_\rho + \frac{\varkappa^2}{\rho^2}\theta_\eta \pm \sqrt{\left(p_\rho + \frac{\varkappa^2}{\rho^2}\theta_\eta\right)^2 + \frac{4\varkappa^2}{\rho^2}\left(p_\eta\theta_\rho - p_\rho\theta_\eta\right)}}{2}$$

with

$$p_\eta\theta_\rho - \theta_\eta p_\rho = -v^2(\varepsilon_{vv}\varepsilon_{\eta\eta} - \varepsilon_{v\eta}^2) < 0, \quad v = 1/\rho.$$

1D case with $\alpha(\rho) = \frac{\varkappa^2}{\rho}$

$$\rho_t + (\rho u)_x = 0,$$

$$(\rho u)_t + \left(\rho u^2 + p\right)_x = 0,$$

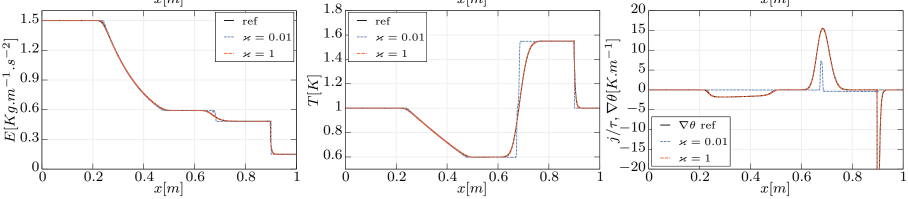
$$(\rho\eta)_t + \left(\rho\eta u + \frac{\varkappa^2}{\rho} j\right)_x = \frac{\varkappa^2}{\rho\theta}\frac{j^2}{\tau} \geq 0,$$

$$j_t + (ju + \theta)_x = -\frac{j}{\tau}.$$

Energy conservation :

$$E_t + \left(Eu + pu + \frac{\varkappa^2}{\rho}\theta j\right)_x = 0, \quad E = \frac{1}{2}\rho u^2 + \rho\varepsilon + \frac{1}{2}\frac{\varkappa^2}{\rho} j^2.$$

Comparison of the Euler model with heat conduction and its hyperbolic version



Rankine-Hugoniot relations

For symplicity, consider again the case $\alpha(\rho) = \dfrac{\varkappa^2}{\rho}$.

$$[\mathcal{M}] = 0, \quad \mathcal{M} = \rho(u - \mathcal{D}), \tag{26a}$$

$$[p + \frac{\mathcal{M}^2}{\rho}] = 0, \tag{26b}$$

$$[\mathcal{M}\left(\frac{\mathcal{M}^2}{2\rho^2} + \varepsilon + \frac{p}{\rho} + \frac{1}{2}\frac{\varkappa^2}{\rho^2}j^2\right) + \frac{\varkappa^2}{\rho}\theta\, j] = 0, \tag{26c}$$

$$[\mathcal{M}\frac{j}{\rho} + \theta] = 0. \tag{26d}$$

Hugoniot curve for hyperbolic heat conductivity

Definition We call *Hugoniot curve* with center (v_0, p_0) the curve in the (v, p)- plane defined as

$$\varepsilon - \varepsilon_0 + \frac{1}{2}(p + p_0)(v - v_0) + \frac{\varkappa^2}{2}\frac{\left(\theta^2 - \theta_0^2\right)(v - v_0)}{(p - p_0)} = 0. \qquad (27)$$

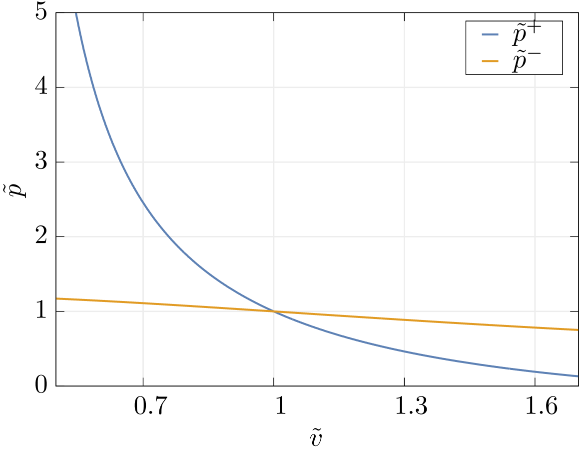


Figure – Two branches of the Hugoniot curve (acoustic ($\tilde{p}^+$) and thermic ($\tilde{p}^-$)).

Oleinik-Liu admissibility criterion

O. Oleinik (scalar conservation law), B. Wendroff (Euler equations with $p_v < 0$), T.- P. Liu (systems of conservation laws).

Oleinik-Liu admissibility criterion

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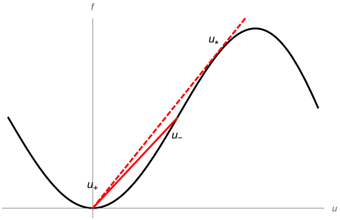
Scalar conservation laws :

$$u_t + f(u)_x = 0.$$

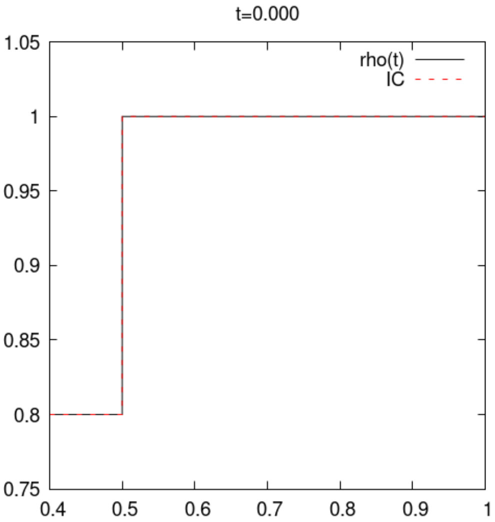


The shock is *admissible* iff

$$D = \frac{f(u_-) - f(u_+)}{u_- - u_+} \geq \frac{f(u) - f(u_+)}{u - u_+}$$

for any u in the interval (u_+, u_-).

Thermic branch : rarefaction shock



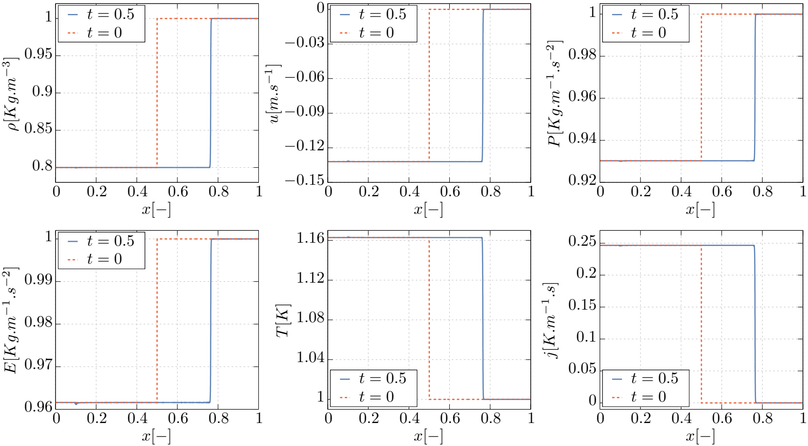


Figure – Rarefaction shock corresponding to the thermic branch

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Do they exist, the rarefaction shocks ?

They were descovered, in particular, in Freon - 13 (a cryogenic fluid having a boiling temperature $T_b = -82^\circ C$) (A. Borisov *et al.*, JFM, 1983).

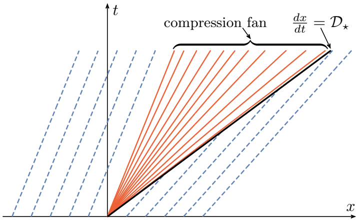


Figure – Composite wave solution : shock–simple wave of compression

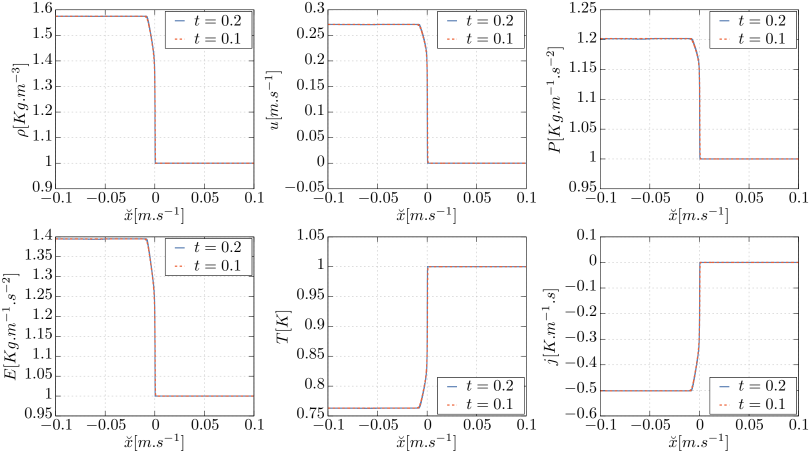


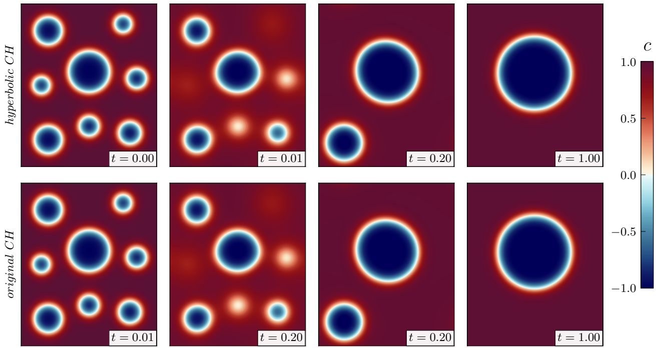
Figure – Composite wave solution as a function of $\breve{x} = \frac{x}{t}$

Cahn-Hilliard equation (F. Dhaouadi, M. Dumbser, SG, 2024)

$$c_t + \mathrm{div}\boldsymbol{J} = 0, \quad \boldsymbol{J} = -\nabla\left(\frac{\delta f}{\delta c}\right), \quad f = \frac{(c^2-1)^2}{4} + \gamma\frac{|\nabla c|^2}{2}. \qquad (28)$$

Lyapunov functional

$$\frac{dF}{dt} \leq 0, \quad F = \int_{\mathcal{D}} f \, d\, D. \qquad (29)$$



Spinodal decomposition

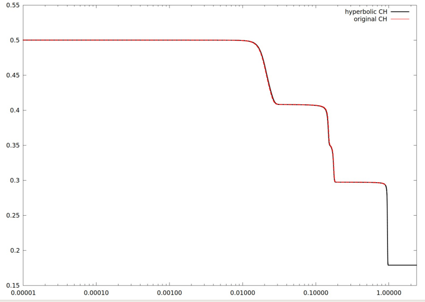


Figure – Behaviour of the dissipation functional in time.

Conclusion

Reversible dispersive models and dissipative models can be reformulated in hyperbolic form with good accuracy, if certain basic properties of the original models are respected (variational formulation, reversibility, entropy inequality, etc.).

References

1. 2024 F Dhaouadi, S Gavrilyuk, Proc. Royal Society A, 480, pp. 0230440.
2. 2023 S Tkachenko, S Gavrilyuk, J Massoni, J. Computational Physics, 111901
3. 2022 F Dhaouadi, S Gavrilyuk, JP Vila, Applied Mathematics and Computation 433, 127378
4. 2022 S Gavrilyuk, H Gouin, Physical Review E 106 (5), 055102
5. 2022 C Besse, S Gavrilyuk, M Kazakova, P Noble, Water Waves 4 (3), 313-343
6. 2022 S. Gavrilyuk and K.–M. Shyue, Nonlinearity, v 35, Issue 3, 1447-1467
7. 2022 S. Gavrilyuk and K.–M. Shyue, Nonlinearity, v. 35, 388–410
8. 2021 S. Busto, M. Dumbser, C. Escalante, N. Favrie, S. Gavrilyuk, J. Scientific Computing (2021) 87 :48 https ://doi.org/10.1007/s10915-021-01429-8
9. 2020 S. L. Gavrilyuk, H. Gouin, Wave Motion 98, 102620
10. 2020 S. Gavrilyuk, B. Nkonga, K.- M. Shyue, L. Truskinovsky, Nonlinearity, 33, N10.
11. 2019, F. Dhaouadi, N. Favrie, S. Gavrilyuk, Studies in Applied Mathematics 142 (3), 336-358
12. 2017 Favrie N., Gavrilyuk S., Nonlinearity 30 (7).