Aline Lefebvre-Lepot - CNRS

Centre Borelli - ENS Paris Saclay
Université Paris-Saclay

Journées scientifiques 2024 RT Terre et Energie
November, 7th, 2024

ANR RheoSuNN

(Partially-)saturated granular media.

Macroscopic, non-brownian entities
Stokes flow

Rheology.

Rheology : ” the branch of physics concerned with
the flow and change of shape of matter ”

[Collins English Dictionary]

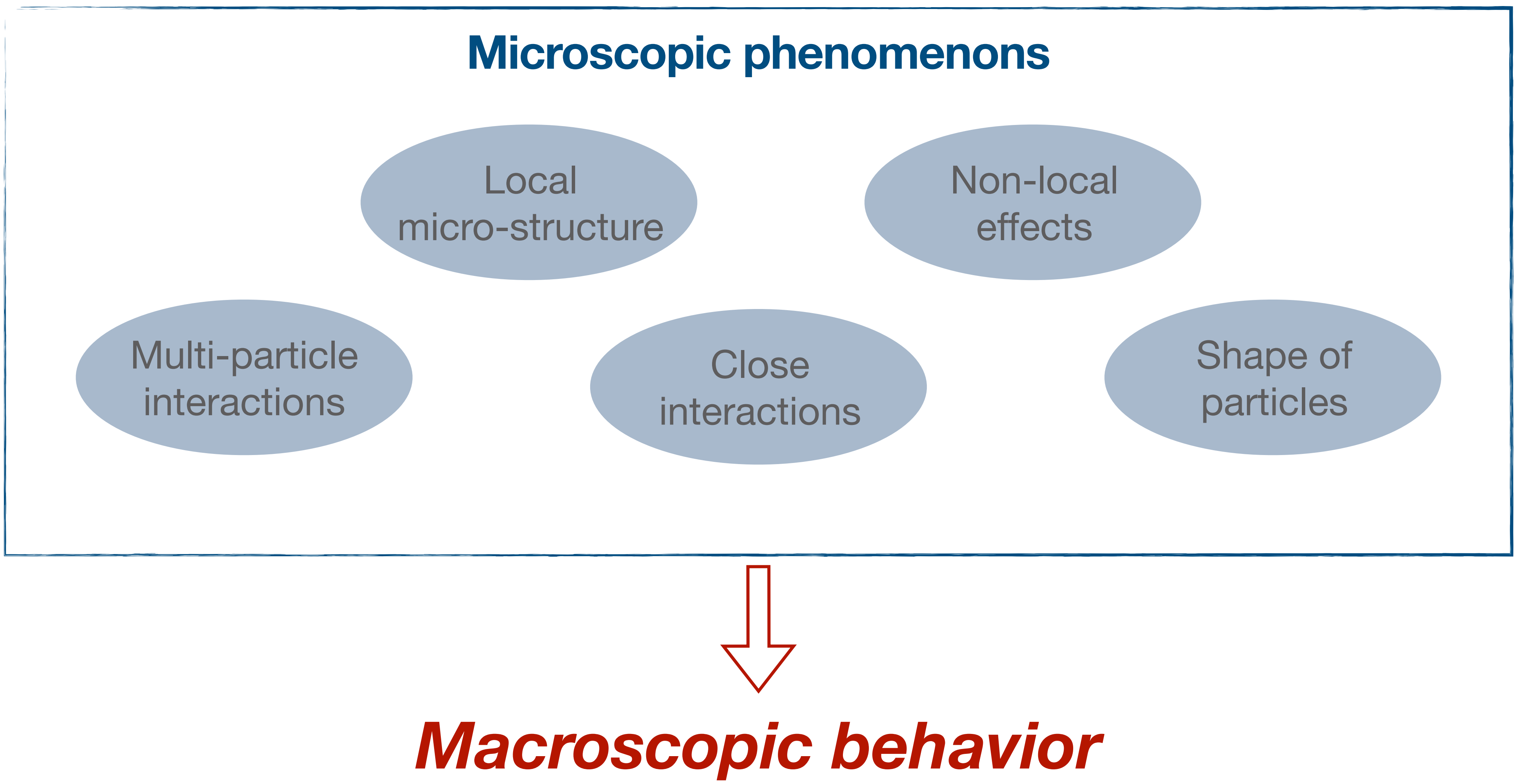
- flow, segregation, mixing, blocking, collapse...

Macroscopic behavior

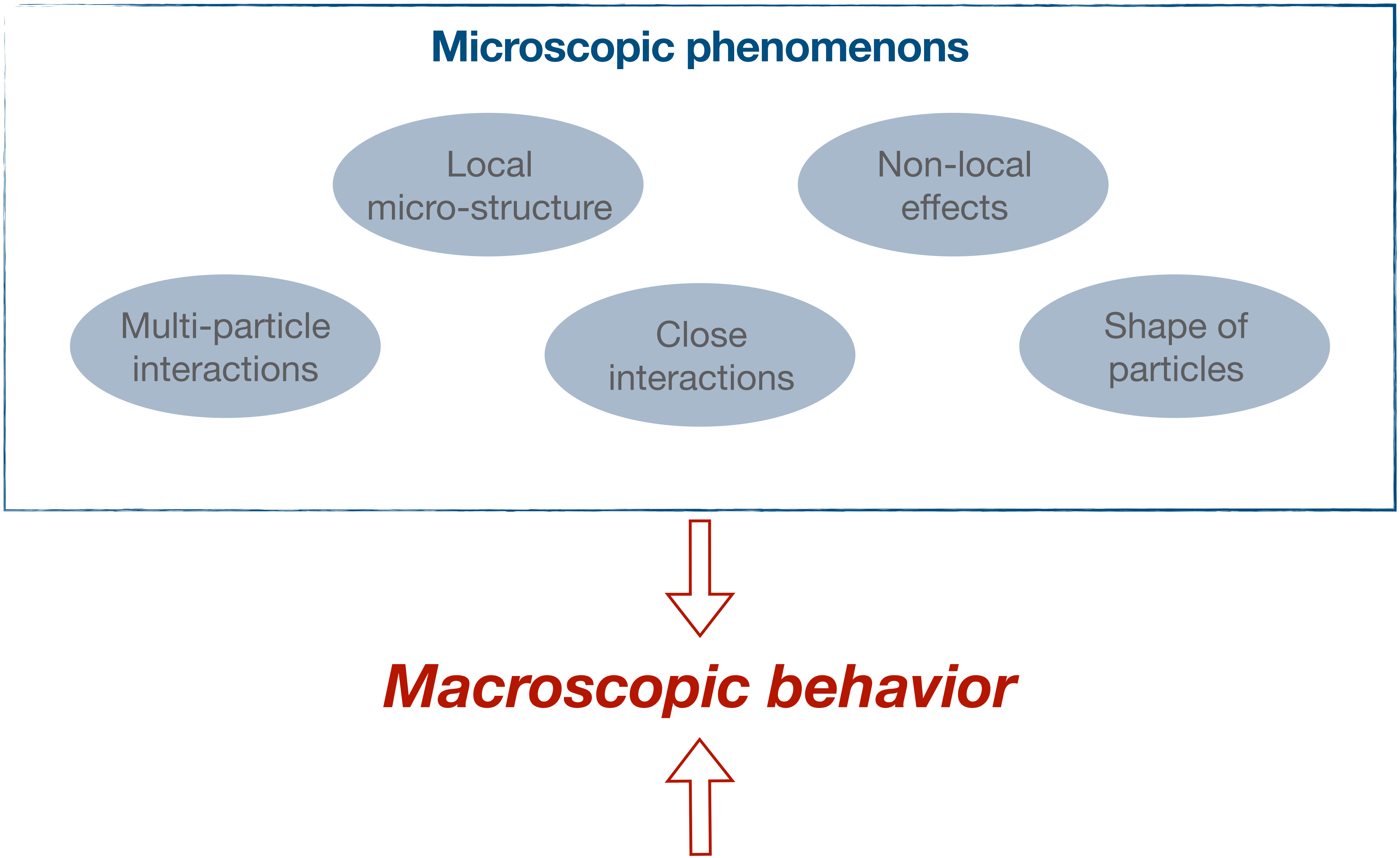
Methodology

Macroscopic behavior

Methodology



Methodology



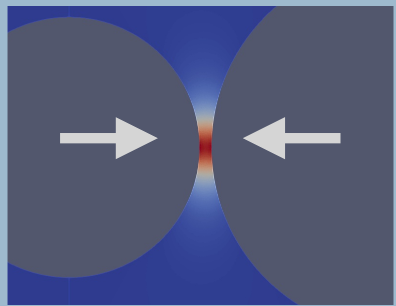
Numerical simulations at the microscopic level

Methodology

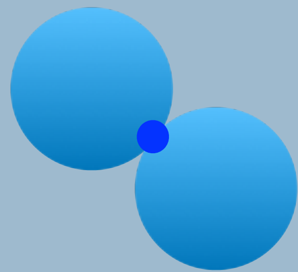
Microscopic phenomenons

Focus on:

Lubrication



Solid contact

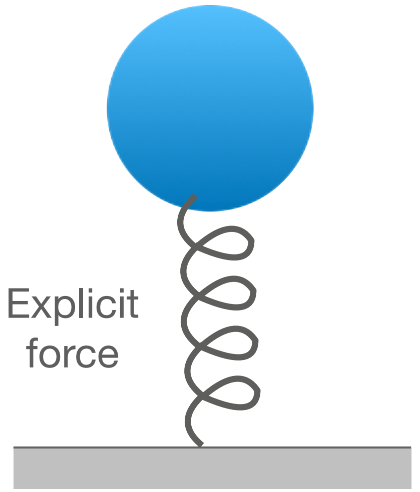


Macroscopic behavior

Numerical simulations at the microscopic level

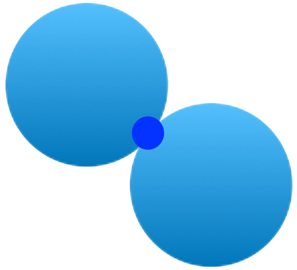
Two class of models.

Explicit penalty force



Continuous penalization method

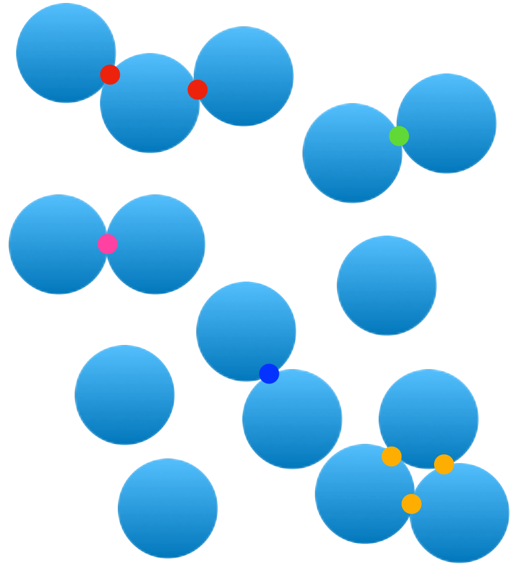
Contact law



Inelastic contacts
Coulomb friction law

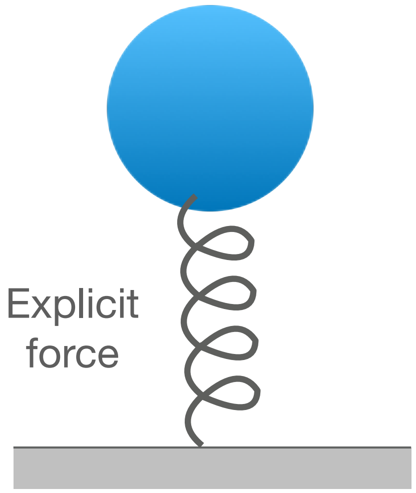
Explicit solution for 2 particles

Multi-contact problem



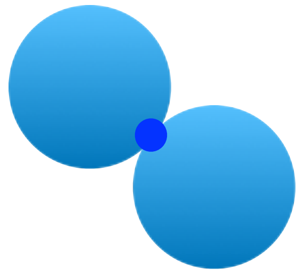
Two class of models.

Explicit penalty force



Continuous penalization method

Contact law



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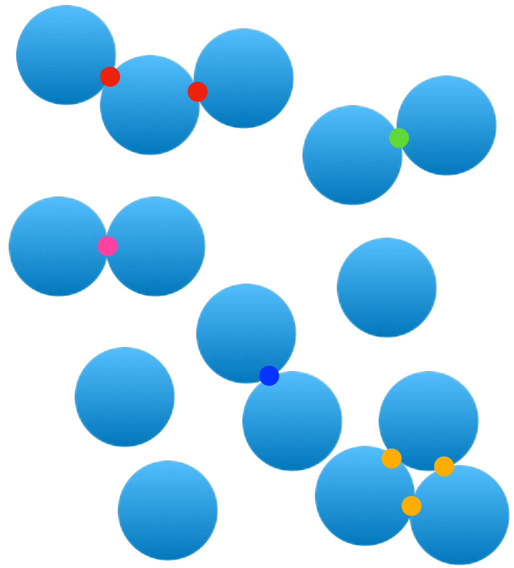
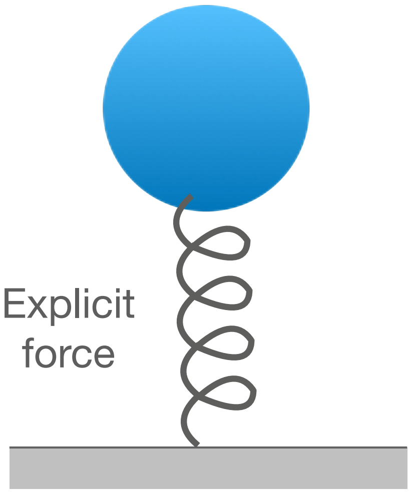


DEM / Molecular Dynamics

[Cundall, Strack, 1979]

Two class of models.

Explicit penalty force

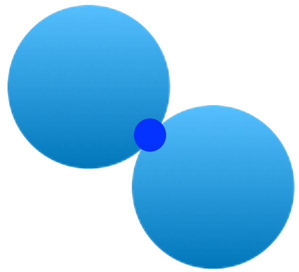


Continuous penalization method

DEM / Molecular Dynamics

[Cundall, Strack, 1979]

Contact law



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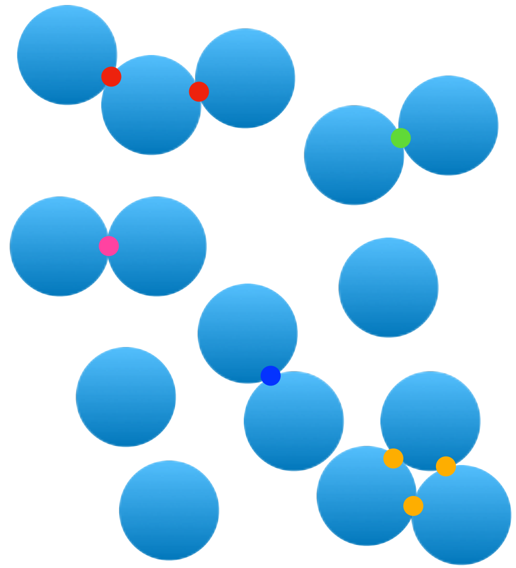
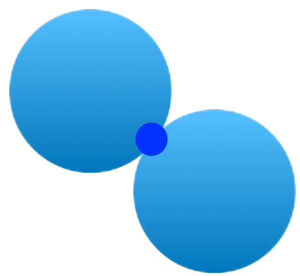


Non-Smooth Contact Dynamics

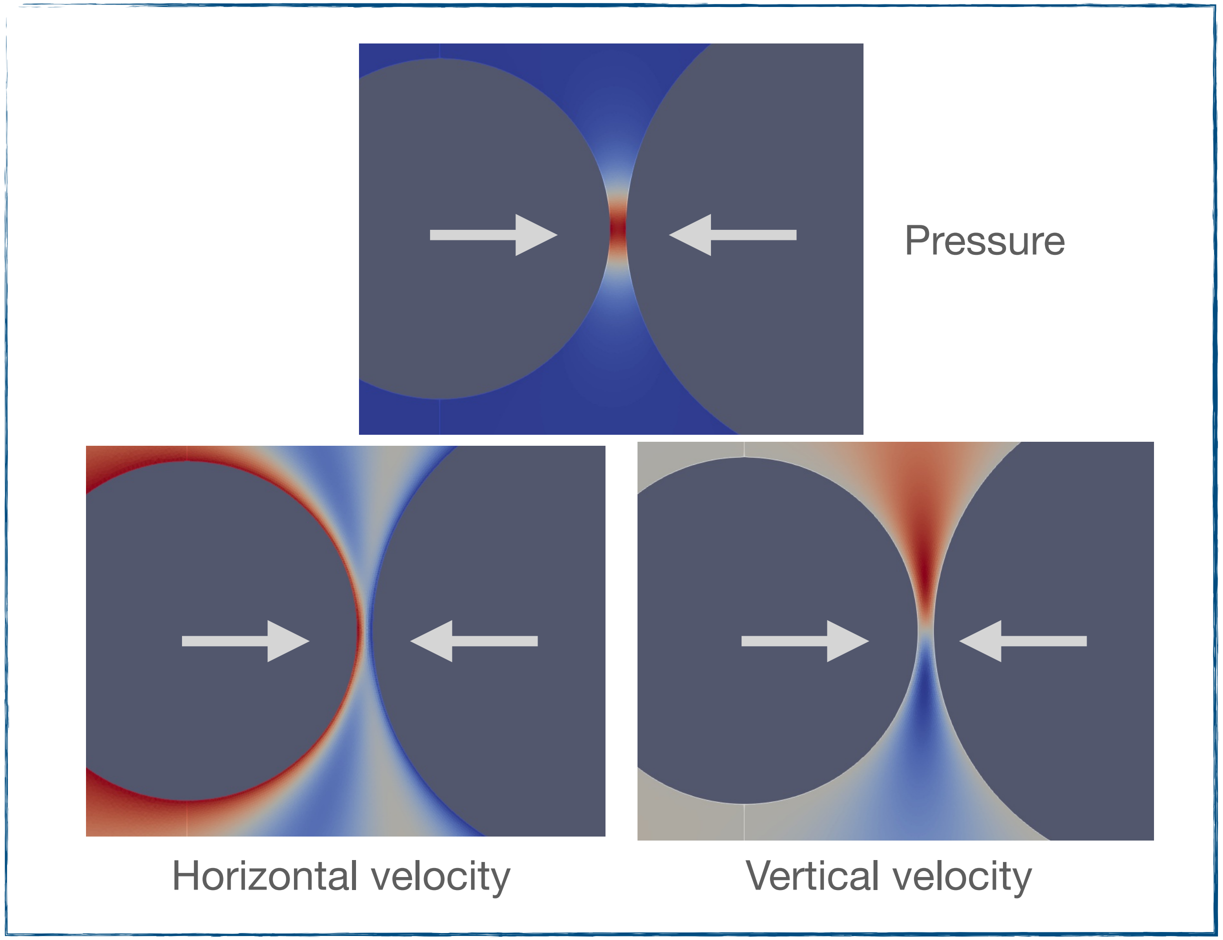
[Moreau, 1988] [Moreau, Jean, 1992]

Contact law



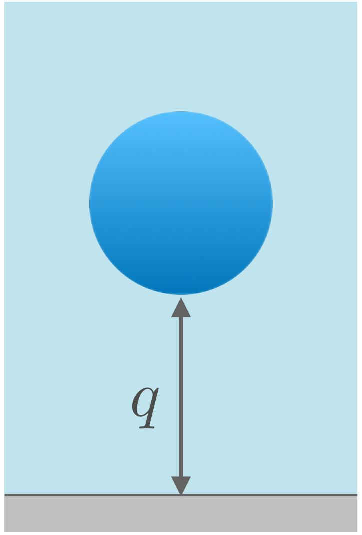
Gluey contact model

Lubrication and Numerical simulations.



A stiff problem.

Lubrication force



$$F_{lub}(q) = -6\pi\mu r^2 \frac{\dot{q}}{q}$$

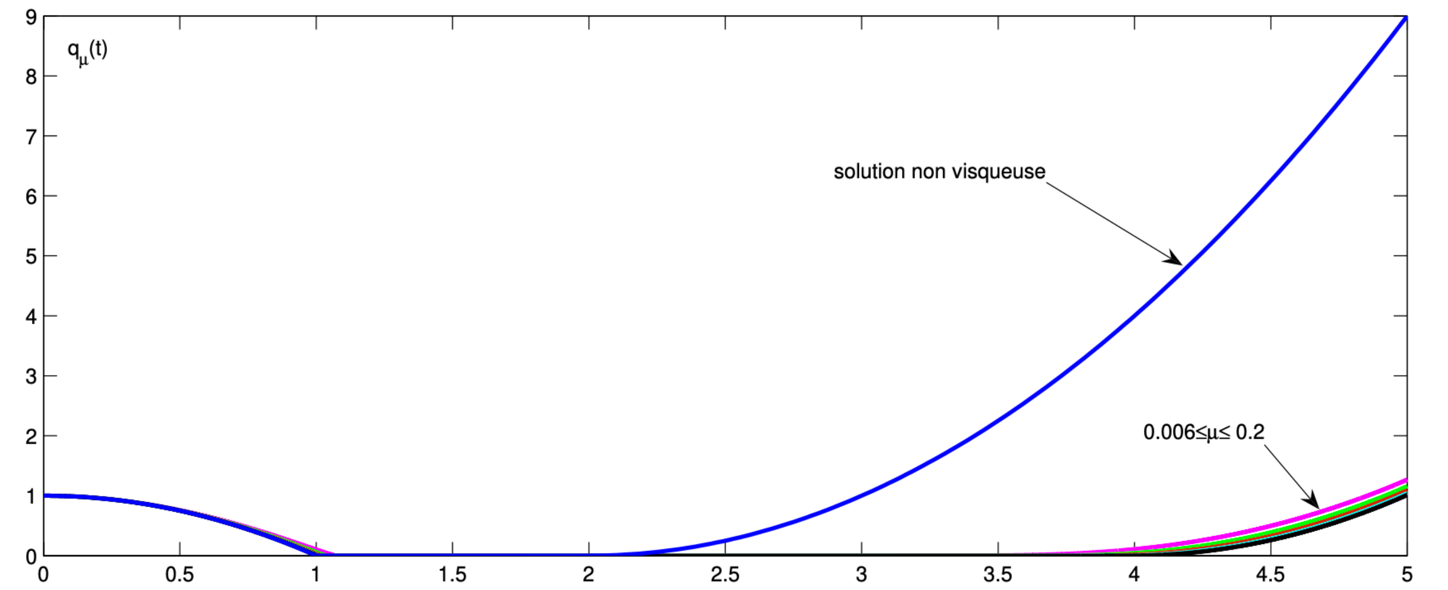
[Cox, 1974]

$$m\ddot{q} = -6\pi\mu r^2 \frac{\dot{q}}{q} + m f^{\text{ext}}$$

- $q(t) > 0$ for any t
- BUT very small…

- Cox, R. G. (1974). The motion of suspended particles almost in contact. *International Journal of Multiphase Flow*, *1*(2), 343-371.

Modeling lubrication: the gluey contact model.

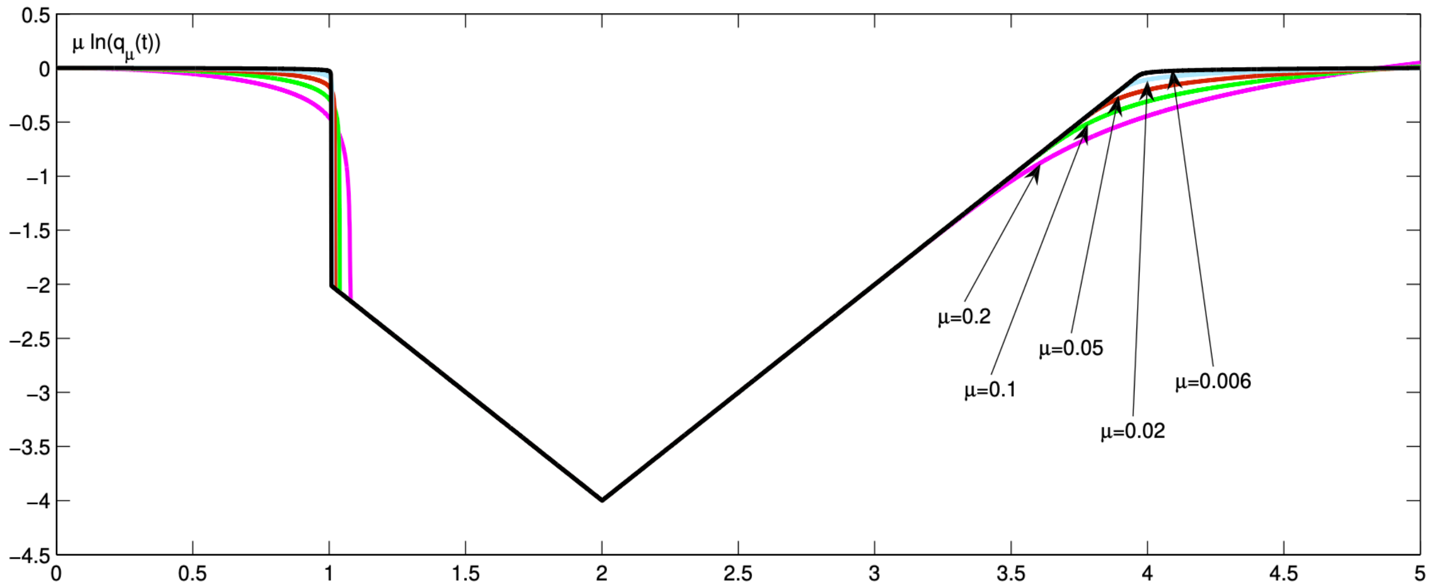


$$f^{\text{ext}} = -2\,\mathbf{1}_{[0,2]} + 2\,\mathbf{1}_{[2,+\infty]}$$

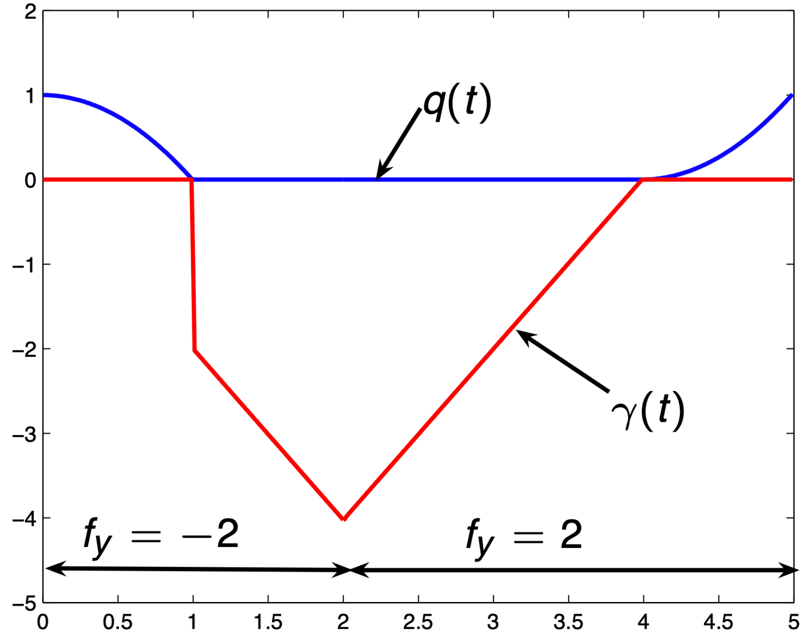
$$\ddot{q}_\mu = -\mu\frac{\dot{q}_\mu}{q_\mu} + f^{\text{ext}}$$

$$\dot{q}_\mu(0) = 0$$

$$q_\mu(0) = 1$$



Modeling lubrication: the gluey contact model.



$$\dot{q}^+ = P_{C_{q,\gamma}} \dot{q}^-$$

$$m\ddot{q} = m f^{\text{ext}} + \lambda_n$$

$$\text{supp}(\lambda_n) \subset \{t, q(t) = 0\}$$

$$\dot{\gamma} = -\lambda_n$$

$$q \geq 0, \gamma \leq 0$$

$$C_{q,\gamma} = \left| \begin{array}{lll} \{0\} & \text{si} & \gamma^- < 0 \\ \mathbb{R}^+ & \text{si} & \gamma^- = 0 \\ & & q = 0 \\ \mathbb{R} & \text{sinon} & \end{array} \right.$$

- B. Maury, A gluey particle model, ESAIM Proceedings, July 2007, Vol.18, 133-142
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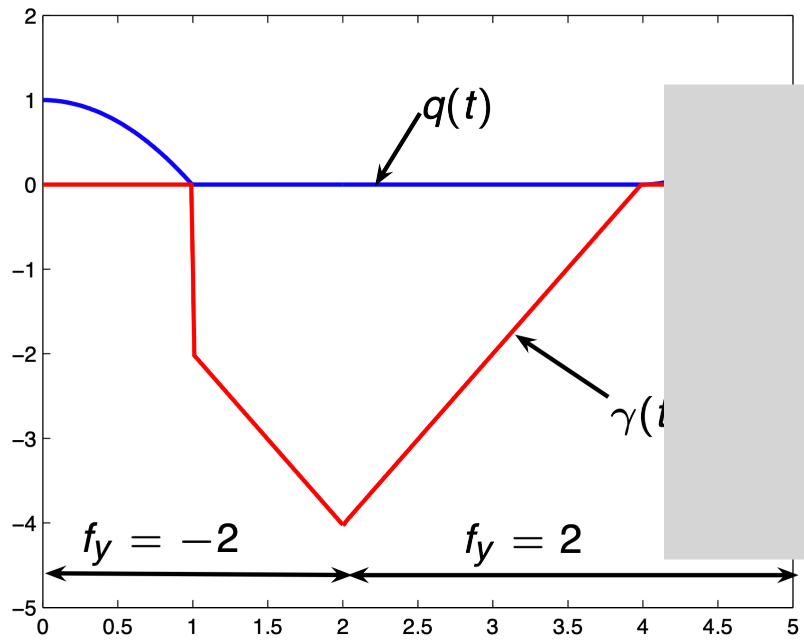

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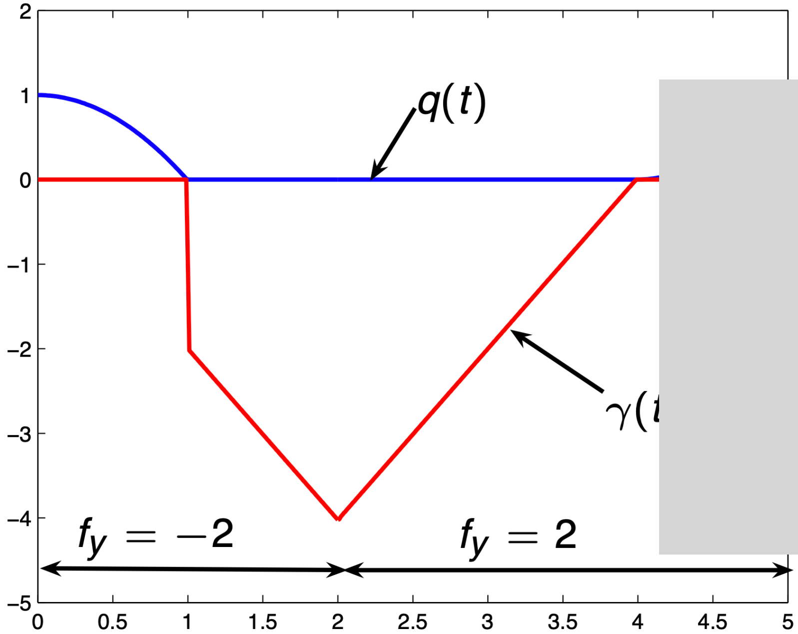

$$\dot{q}^+ = P_{\mathbb{R}^+}\dot{q}^- = 0$$

$$\dot{q}^+ = P_{C_{q,\gamma}}\dot{q}^-$$

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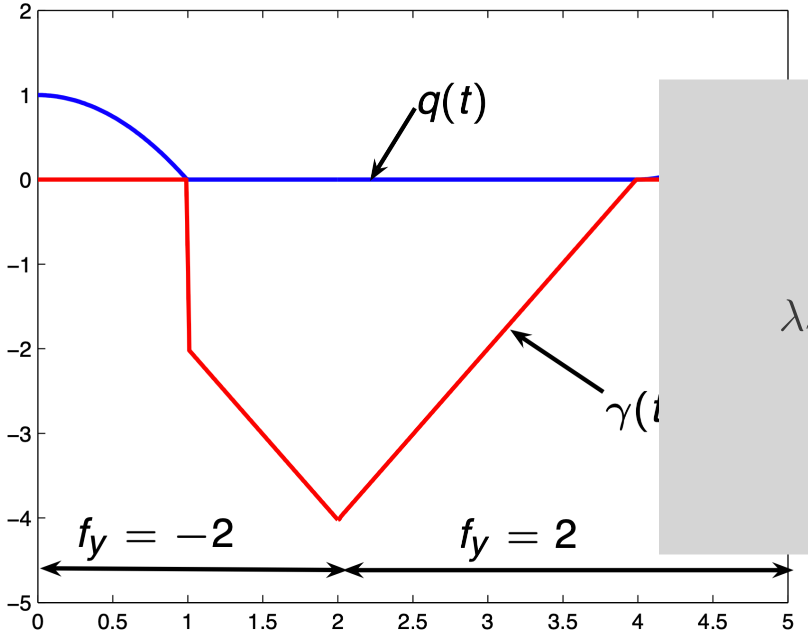

$$\dot{q}^+ = P_{\mathbb{R}^+}\dot{q}^- = 0$$

$$\lambda_n = m(\dot{q}^+ - \dot{q}^-) = -m\dot{q}^-$$

$$\dot{q}^+ = P_{C_{q,\gamma}}\dot{q}^-$$

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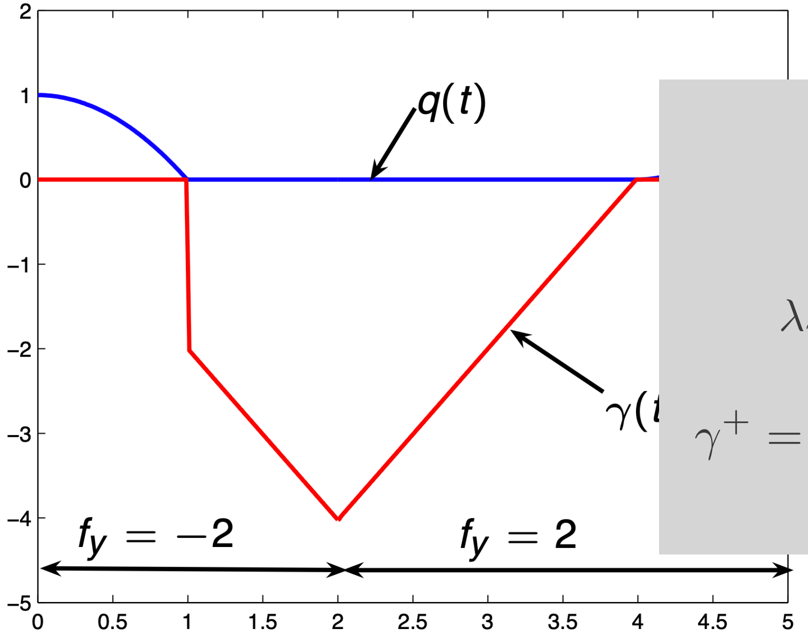

$$\dot{q}^+ = P_{\mathbb{R}^+}\dot{q}^- = 0$$

$$\lambda_n = m(\dot{q}^+ - \dot{q}^-) = -m\dot{q}^-$$

$$\gamma^+ = \gamma^+ - \gamma^- = -\lambda_n = m\dot{q}^- < 0$$

$$\dot{q}^+ = P_{C_{q,\gamma}}\dot{q}^-$$

$$m\ddot{q} = mf^{\text{ext}} + \lambda_n$$

$$\text{supp}(\lambda_n) \subset \{t, q(t) = 0\}$$

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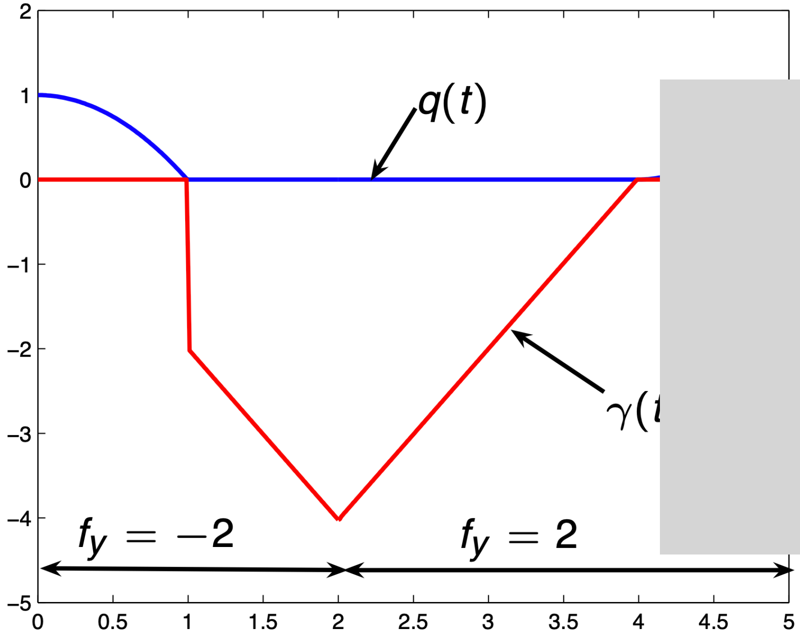

$\gamma < 0 \Rightarrow q \equiv 0$

$$\dot{q}^+ = P_{C_{q,\gamma}} \dot{q}^-$$

$$m\ddot{q} = mf^{\text{ext}} + \lambda_n$$

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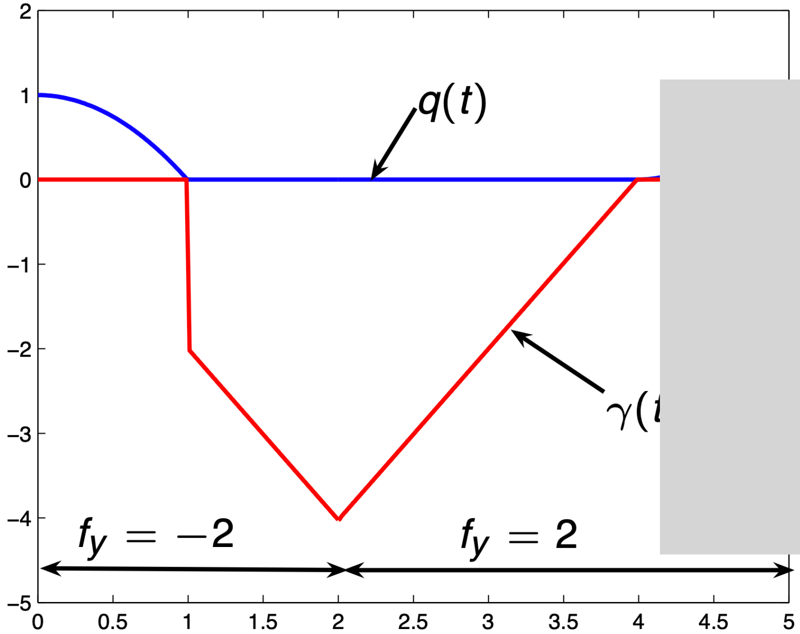

$$\gamma < 0 \Rightarrow q \equiv 0$$

$$\lambda_n = -m f^{\mathrm{ext}}$$

$$\dot{q}^+ = P_{C_{q,\gamma}} \dot{q}^-$$

$$m\ddot{q} = m f^{\mathrm{ext}} + \lambda_n$$

$$\mathrm{supp}(\lambda_n) \subset \{t, q(t) = 0\}$$

$$\dot{\gamma} = -\lambda_n$$

$$q \geq 0, \gamma \leq 0$$

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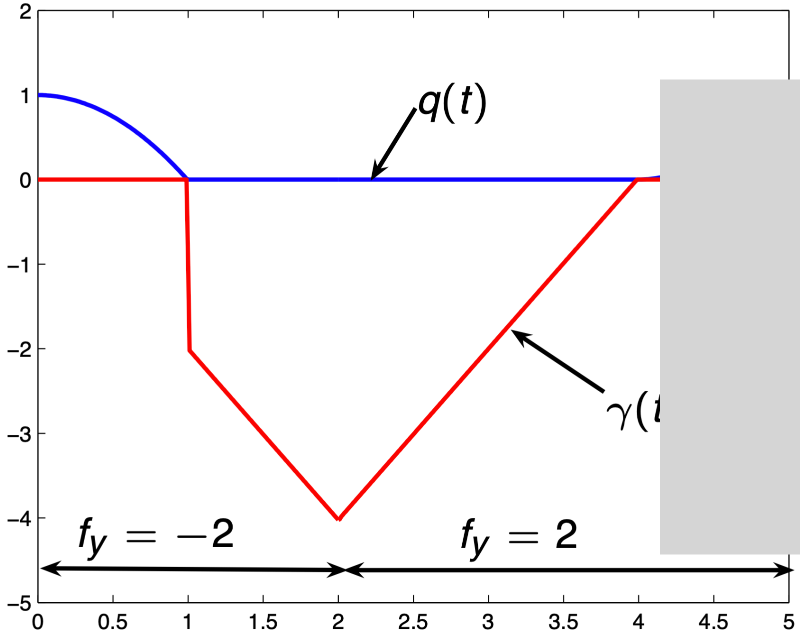

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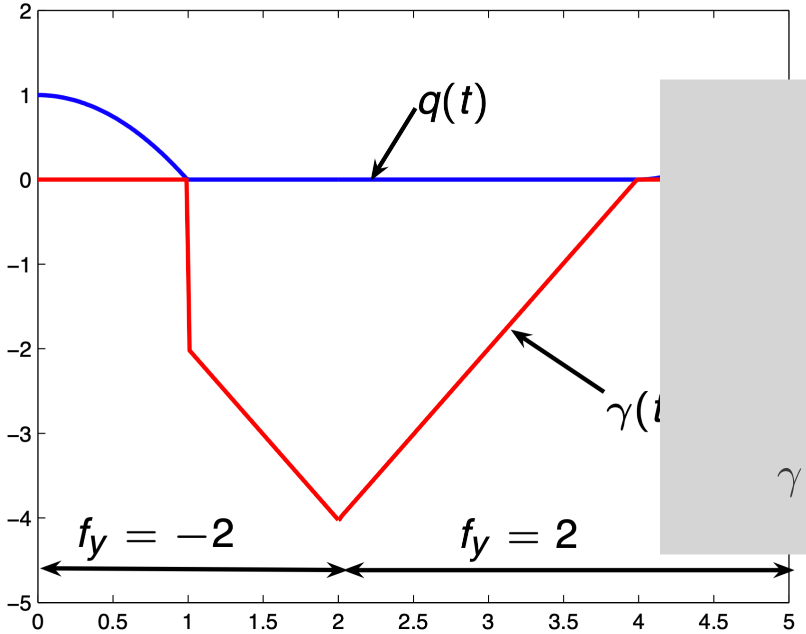

$$\gamma < 0 \Rightarrow q \equiv 0$$

$$\lambda_n = -m f^{\text{ext}}$$

$$\dot\gamma = -\lambda_n = m f^{\text{ext}}$$

$$\gamma(t) = m\dot q^-(t_c) + \int_{t_c}^{t} m f^{\text{ext}}$$

$$\dot q^+ = P_{C_{q,\gamma}} \dot q^-$$

$$m\ddot q = m f^{\text{ext}} + \lambda_n$$

$$\text{supp}(\lambda_n) \subset \{t, q(t) = 0\}$$

$$\dot\gamma = -\lambda_n$$

$$q \geq 0, \gamma \leq 0$$

$$C_{q,\gamma} = \left| \begin{array}{lll} \{0\} & \text{si} & \gamma^- < 0 \\ \mathbb{R}^+ & \text{si} & \gamma^- = 0 \\ & & q = 0 \\ \mathbb{R} & \text{sinon} & \end{array} \right.$$

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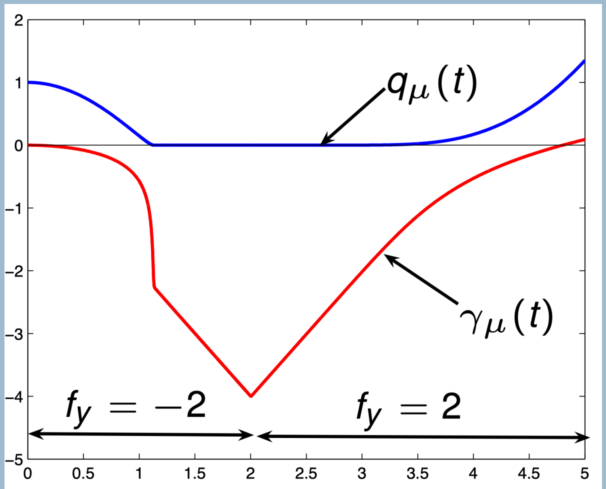
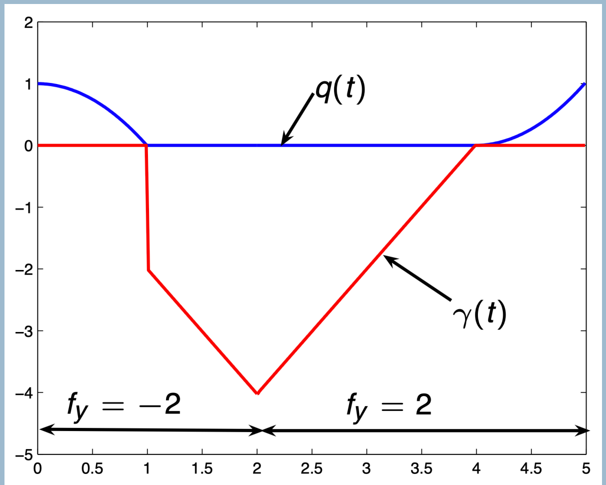
Convergence result: first order equations.



$$m\dot{q}_\mu(t) + \gamma_\mu(t) = m\dot{q}(0) + \gamma_\mu(0) + m\int_0^t f^{\text{ext}}(s)ds$$

$$\gamma_\mu(t) = \mu \ln(q_\mu(t))$$

$$\mu \to 0$$



$$m\dot{q}(t) + \gamma(t) = m\dot{q}(0) + m\int_0^t f^{\text{ext}}(s)ds$$

$$q \geq 0, \quad \gamma \leq 0, \quad q\,\gamma = 0$$

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Numerical scheme

Contact law: $\dot{q}^{+} = P_{C_{q,\gamma}} \dot{q}^{-}$

- Fully implicit algorithms

$$\bar{u}^{k+1} = u^{k} + \Delta t\, f^{\mathrm{ext},k+1}$$

$$u^{k+1} = P_{m,C_{q^k}}\left(\bar{u}^{k+1}\right)$$

- **Implicit** admissible velocity space

$$\begin{aligned} C_{q^k} = \{ v \in \mathbb{R} \,/\, & q^{k+1} = q^k + \Delta t\, v \geq 0 \\ & q^{k+1} = q^k + \Delta t\, v \leq 0, \quad \gamma^k < 0 \} \end{aligned}$$

- Anitescu, M., & Hart, G. D. (2004). A constraint-stabilized time-stepping approach for rigid multibody dynamics with joints, contact and friction. *International Journal for Numerical Methods in Engineering*, *60*(14), 2335-2371.
- B. Maury. A time-stepping scheme for inelastic collisions. *Numerische Mathematik*, 102(4):649–679, 2006.
- A. Lefebvre, Numerical simulation of gluey particles, M2AN, 43:53-80 (2009)

Numerical scheme

$$\left| \begin{array}{l} u \in K \\ J(u) = \min_{v \in K} J(v) \end{array} \right.$$

Primal problem

- Fully implicit algorithms

$$\bar{u}^{k+1} = u^k + \Delta t\, f^{\text{ext},k+1}$$

$$u^{k+1} = P_{m,C_{q^k}}\left(\bar{u}^{k+1}\right)$$

Convex constrained optimization

- Implicit admissible velocity space

$$C_{q^k} = \{v \in \mathbb{R} \,/\, q^{k+1} = q^k + \Delta t\, v \geq 0$$

$$q^{k+1} = q^k + \Delta t\, v \leq 0, \quad \gamma^k < 0\}$$

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Numerical scheme

$$\nabla J(u) = -\lambda \nabla g(u)$$
$$g(u) \leq 0, \quad \lambda \geq 0$$
$$g(u)\lambda = 0$$

Optimality conditions

- Fully implicit algorithms - Convergence result

Model

$$\dot{q}^+ = P_{C_{q,\gamma}} \dot{q}^-$$

$$m\ddot{q} = m f^{\text{ext}} + \lambda_n$$

$$q \geq 0$$

$$\text{supp}(\lambda_n) \subset \{t, q(t) = 0\}$$

$$\dot{\gamma} = -\lambda_n \ , \gamma \leq 0$$

Algorithm

$$u^{k+1} = P_{m, C_{q^k}} \left(\bar{u}^{k+1}\right)$$

$$m \frac{u^{k+1} - u^k}{\Delta t} = f^{\text{ext}}(t^k, q^k) + \lambda_+^{k+1} - \lambda_-^{k+1}$$

$$q^k + \Delta t \, u^{k+1} \geq 0, \quad \lambda_+^{k+1} \geq 0 \quad \lambda_-^{k+1} \geq 0$$
$$\left(q^k + \Delta t \, u^{k+1}\right) \lambda_+^{k+1} = 0$$
$$\left(q^k + \Delta t \, u^{k+1}\right) \lambda_-^{k+1} = 0$$

$$\gamma^{k+1} = \min(0, \gamma^k - \Delta t \, \lambda^{k+1})$$

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Numerical scheme

$$\left| \begin{array}{l} \lambda \in F \\ E(\lambda) = \min_{\mu \in F} E(\mu) \end{array} \right.$$

Dual problem

- Fully implicit algorithms - Dual problem

$$\left| \begin{array}{l} (\lambda_+, \lambda_-) \in F \\ E(\lambda_+, \lambda_-) = \min_{(\mu_+, \mu_-) \in F} E(\mu_+, \mu_-) \end{array} \right.$$

Discrete energy

$$F = \mathbb{R}^+ \times \mathbb{R}^+$$

$$E(\mu_+, \mu_-) = (\mu_+ - \mu_-) \left(q^k + \Delta t \, \frac{u_\mu + \bar{u}^{k+1}}{2} \right)$$

where $$m \frac{u_\mu - u^k}{\Delta t} = f^{\text{ext}}(t^k, q^k) + \mu_+ - \mu_-$$

- On convex numerical schemes for inelastic contacts with friction, with H. Bloch, ESAIM:Proc 75 (2023), pp. 24-59
- M. Frémond. Non-Smooth Thermomechanics. Springer Berlin Heidelberg, Berlin, Heidelberg, 2002

$$E(\mu) = \mu \left(\frac{u^+ + u^-}{2} \right)$$

Numerical scheme

$$\left| \begin{array}{l} \lambda \in F \\ E(\lambda) = \min_{\mu \in F} E(\mu) \end{array} \right.$$

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$$E(\mu_+, \mu_-) = (\mu_+ - \mu_-) \left(q^k + \Delta t \, \frac{u_\mu + \bar{u}^{k+1}}{2} \right)$$

Discrete energy

- Can be solved using Projected Gradient Algorithms

$$E(\mu) = \mu \left(\frac{u^+ + u^-}{2} \right)$$

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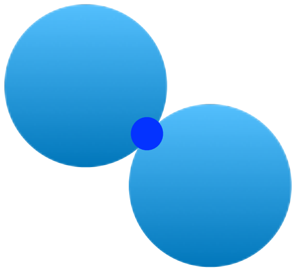
- M. Frémond. Non-Smooth Thermomechanics. Springer Berlin Heidelberg, Berlin, Heidelberg, 2002

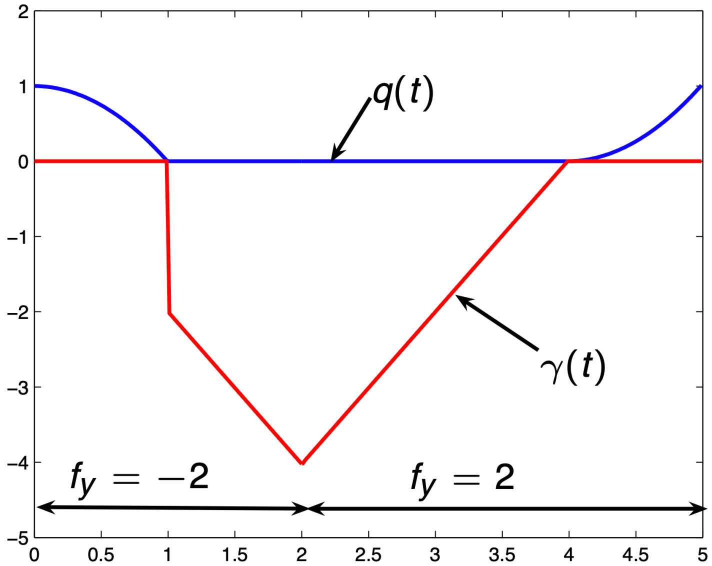
Contact law

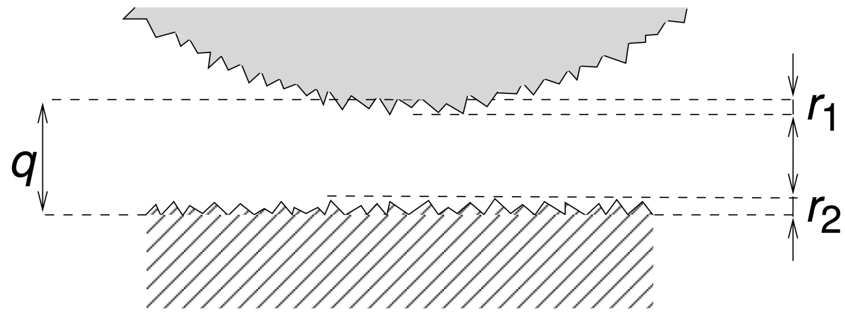


Gluey contact model
with inelastic contact

[PhD Quentin Houssier]

Accounting for solid contact

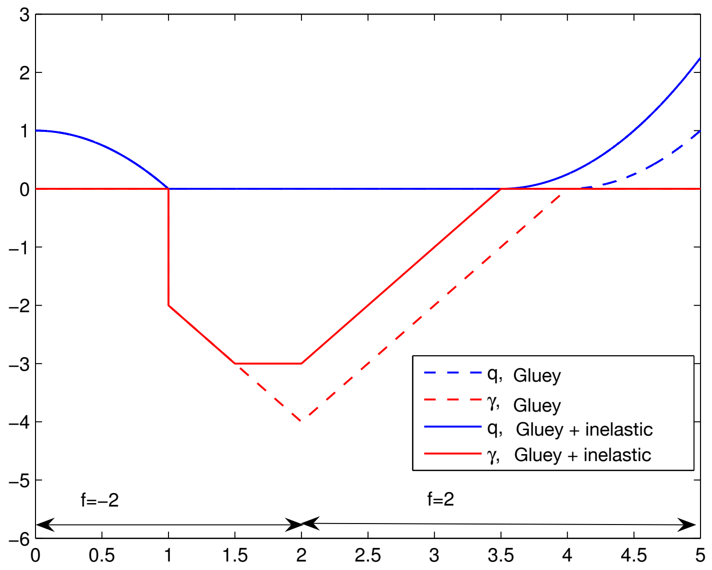


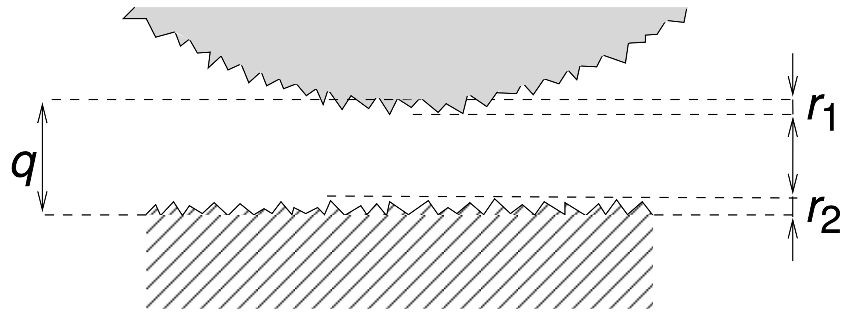


- Roughness $\Longrightarrow$ contact
- Model : solid contact for $q = r_1 + r_2$

- Vinogradova, Yakubov, Surface Roughness and Hydrodynamic Boundary Conditions, PRE 73(4) 2006
- Gérard-Varet, Hillairet, Computation of the drag force on a sphere close to a wall: the roughness issue, ESAIM M2AN, 46 (2012)

Accounting for solid contact





- Roughness $\Longrightarrow$ contact

$$\gamma \approx \gamma_\mu(t) = \mu \ln(q_\mu(t))$$

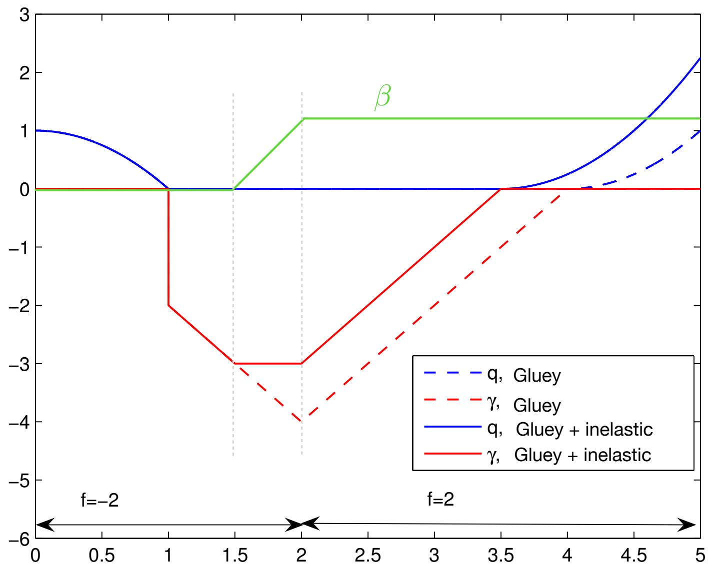
- Model : solid contact for $q = r_1 + r_2$

$\Longrightarrow$ Threshold for γ : $\gamma \geq \gamma^* = \mu \ln(r_1 + r_2)$

$\Longrightarrow$ Easy to implement!

- Vinogradova, Yakubov, Surface Roughness and Hydrodynamic Boundary Conditions, PRE 73(4) 2006
- Gérard-Varet, Hillairet, Computation of the drag force on a sphere close to a wall: the roughness issue, ESAIM M2AN, 46 (2012)

Accounting for solid contact



Model

$$m\dot{q}(t) + \gamma(t) + \beta(t) = m\dot{q}(0) + m\int_0^t f^{\text{ext}}(s)ds$$

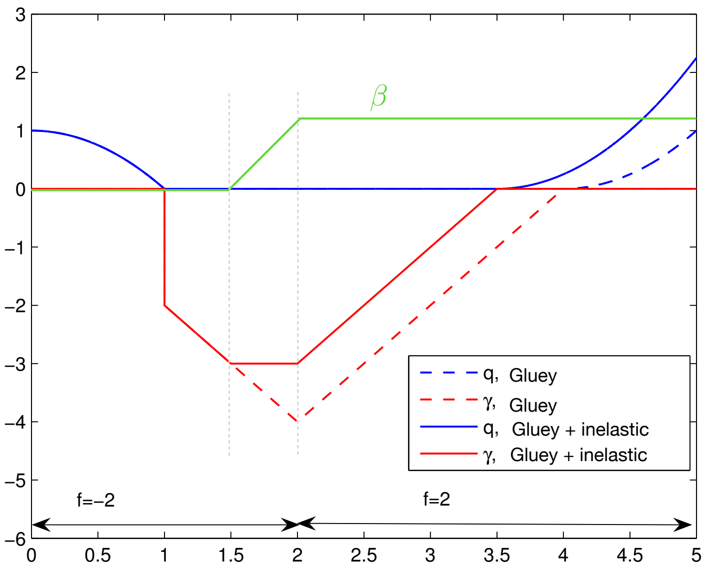
$$q \geq 0, \quad \gamma \leq 0, \quad q\,\gamma = 0$$

$$\gamma \geq \gamma^*, \quad \dot{\beta} \geq 0$$

$$\text{supp}(\dot{\beta}) \subset \{t, \gamma(t^+) = \gamma^*\}$$

[PhD Quentin Houssier]

Accounting for solid contact



- Regularity issues
- Convergence of the numerical scheme
- Existence of solutions

Model

$$m\dot{q}(t) + \gamma(t) + \beta(t) = m\dot{q}(0) + m\int_0^t f^{\text{ext}}(s)ds$$

$$q \geq 0, \quad \gamma \leq 0, \quad q\,\gamma = 0$$

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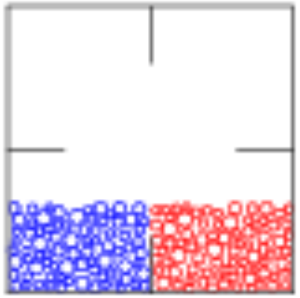
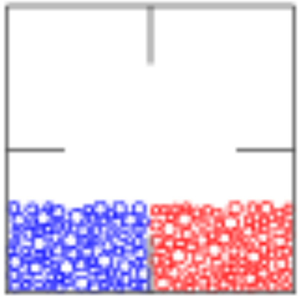
[PhD Quentin Houssier]

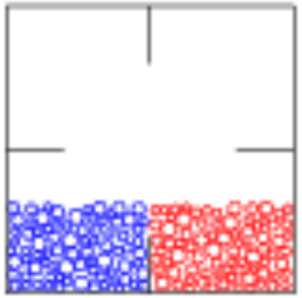
Accounting for solid contact



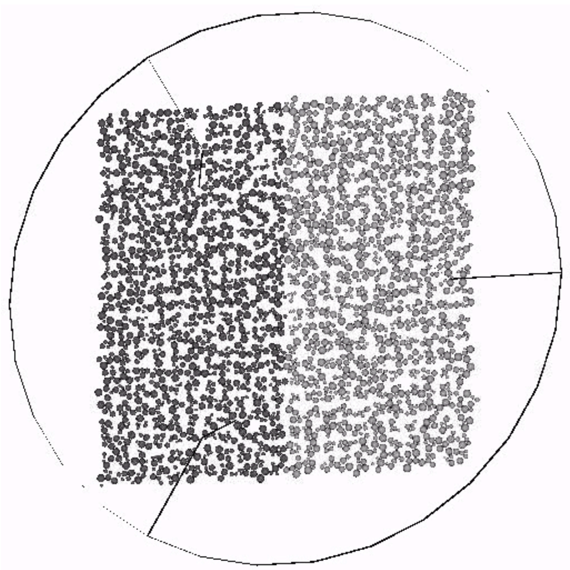
Inelastic contact



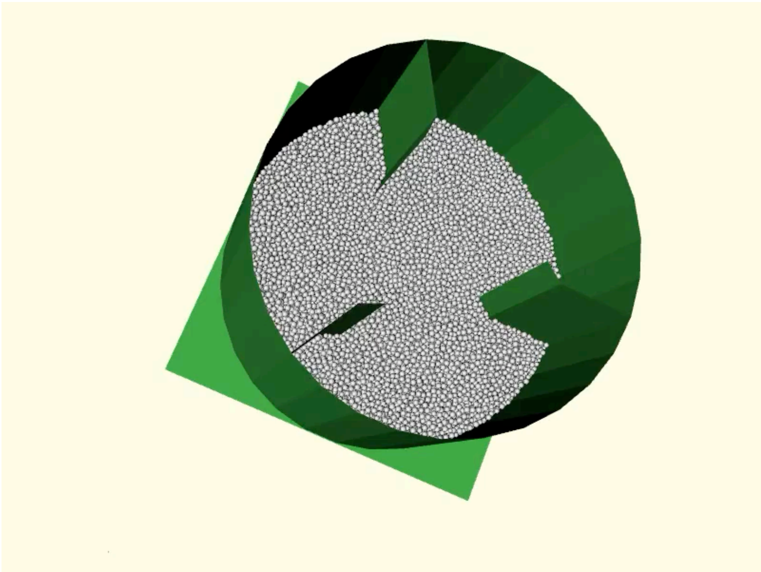
Gluey + Inelastic contact



Gluey contact

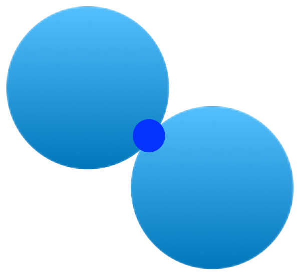


Gluey + Inelastic contact



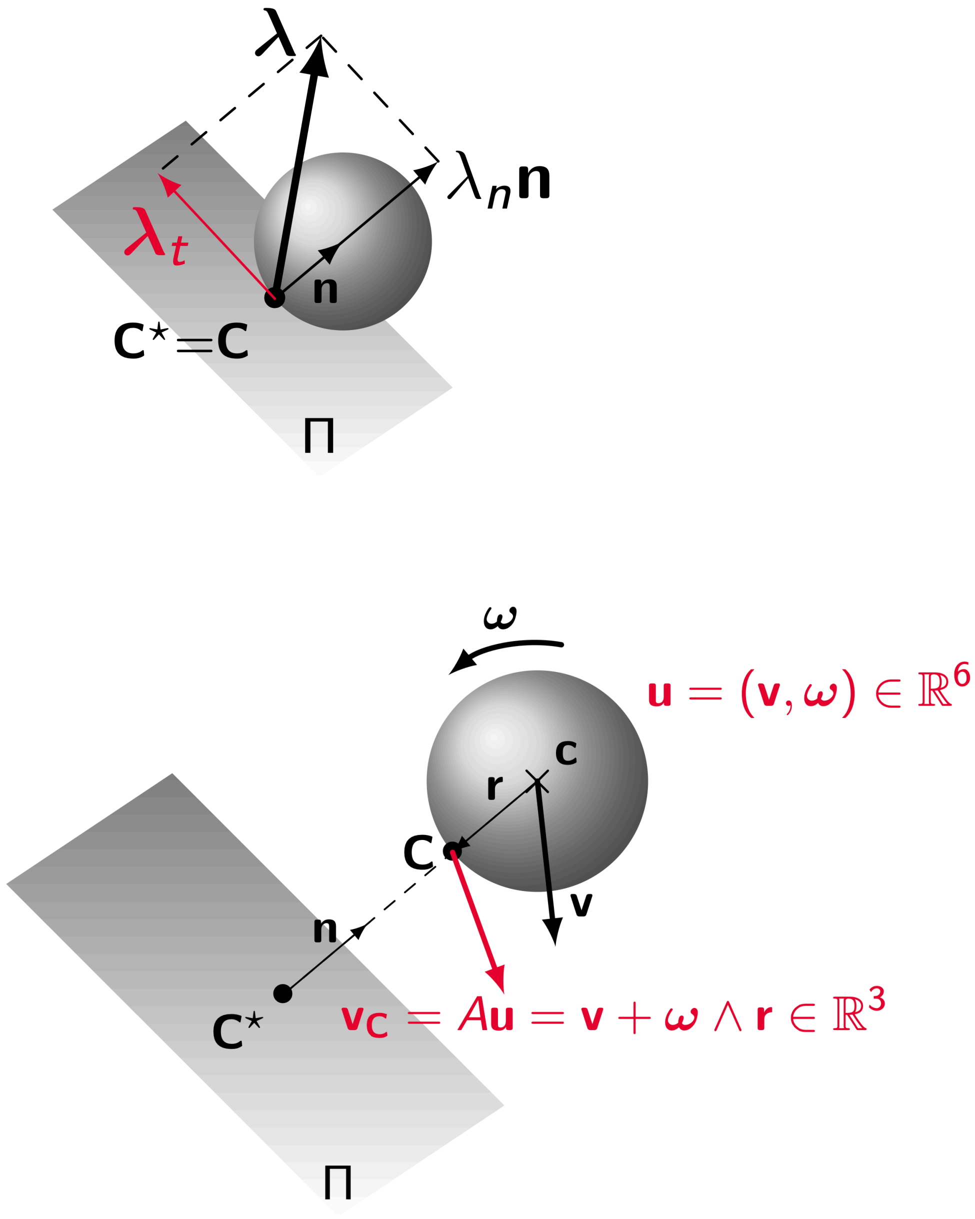
Gluey + Inelastic contact

Contact law



Inelastic contacts
Coulomb friction law

Modeling friction.



- Contact force

$$\boldsymbol{\lambda} = \lambda_n \mathbf{n} + \boldsymbol{\lambda}_t \in \mathbb{R}^3$$

- If $\mathrm{T}\mathbf{v_C}^+ = 0$ (no slip)

$$|\boldsymbol{\lambda}_t| \leq \mu \lambda_n$$

- If $\mathrm{T}\mathbf{v_C}^+ \neq 0$ (sliding motion)

$$\boldsymbol{\lambda}_t = -\mu \lambda_n \frac{\mathrm{T}\mathbf{v_C}^+}{|\mathrm{T}\mathbf{v_C}^+|}$$

Modeling friction.

$$M \frac{d\mathbf{u}}{dt} = \mathbf{f}^{\text{ext}} + A^T(\lambda_n \mathbf{n} + \boldsymbol{\lambda}_t) \quad \text{(FPD)}$$

$$D(\mathbf{c}) \geq 0, \quad \lambda_n \geq 0, \quad D(\mathbf{c})\lambda_n = 0 \quad \text{(Norm. Cont.)}$$

$$\mathbf{u}^+ = P_{C_\mathbf{c}} \mathbf{u}^-$$

If $\mathrm{T}A\mathbf{u}^+ = 0$ (no slip), $|\boldsymbol{\lambda}_t| \leq \mu \lambda_n$ (Tang. Cont.)

If $\mathrm{T}A\mathbf{u}^+ \neq 0$ (sliding motion), $\boldsymbol{\lambda}_t = -\mu \lambda_n \frac{\mathrm{T}\mathbf{v_C}^+}{|\mathrm{T}\mathbf{v_C}^+|}$

$$\left[\mathbf{u} = (\mathbf{v}, \boldsymbol{\omega}), \quad A\mathbf{u} = \mathbf{v} + \boldsymbol{\omega} \wedge \mathbf{r}, \quad A^T \boldsymbol{\lambda} = (\boldsymbol{\lambda}, \mathbf{r} \wedge \boldsymbol{\lambda}) \in \mathbb{R}^6\right]$$

Moreau, J.-J. "Unilateral contact and dry friction in finite freedom dynamics." *Nonsmooth mechanics and Applications* (1988): 1-82.

Founding NSCD algorithms

$$M\frac{\mathbf{u}^{k+1}-\mathbf{u}^{k}}{\Delta t} = \mathbf{f}^{e} + A^{k,T}(\lambda_n \mathbf{n}^k + \boldsymbol{\lambda}_t) \quad \text{(FPD)}$$

$$\lambda_n \geq 0 \quad \text{(Norm. Cont.)}$$

$$D^k + \Delta t\, \nabla D^k \cdot \mathbf{v}^{k+1} \geq 0$$

$$(D^k + \Delta t\, \nabla D^k \cdot \mathbf{v}^{k+1})\lambda_n = 0$$

If $\mathrm{T}^k A^k \mathbf{u}^{k+1} = 0 \qquad |\boldsymbol{\lambda}_t| \leq \mu\lambda_n$ (Tang. Cont.)

If $\mathrm{T}^k A^k \mathbf{u}^{k+1} \neq 0 \qquad \boldsymbol{\lambda}_t = -\mu\lambda_n \dfrac{\mathrm{T}^k A^k \mathbf{u}^{k+1}}{|\mathrm{T}^k A^k \mathbf{u}^{k+1}|}$

- Jean, M. (1999). The non-smooth contact dynamics method. *Computer methods in applied mechanics and engineering*, *177*(3-4), 235-257.
- Dubois, F., & Jean, M. (2006). The non smooth contact dynamic method: recent LMGC90 software developments and application. *Analysis and simulation of contact problems*, 375-378.

Founding NSCD algorithms

Non-convex problem

$$M\frac{\mathbf{u}^{k+1}-\mathbf{u}^{k}}{\Delta t}=\mathbf{f}^{e}+A^{k,T}(\lambda_n\mathbf{n}^{k}+\boldsymbol{\lambda}_t)$$ (FPD)

$$\lambda_n \geq 0$$ (Norm. Cont.)

$$D^k+\Delta t\,\nabla D^k\cdot\mathbf{v}^{k+1}\geq 0$$

$$(D^k+\Delta t\,\nabla D^k\cdot\mathbf{v}^{k+1})\lambda_n=0$$

If $\mathrm{T}^k A^k\mathbf{u}^{k+1}=0 \qquad |\boldsymbol{\lambda}_t|\leq\mu\lambda_n$ (Tang. Cont.)

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Convex optimization for friction?

$$\left|\begin{array}{l}\lambda \in F \\ E(\lambda) = \min_{\mu \in F} E(\mu)\end{array}\right.$$

Dual problem

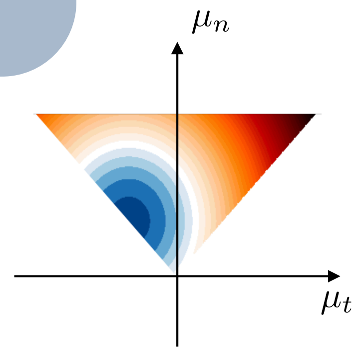
Discrete energy

$$\left|\begin{array}{l}\boldsymbol{\lambda} \in \mathbf{F} \\ E(\boldsymbol{\lambda}) = \min_{\boldsymbol{\mu} \in \mathbf{F}} E(\boldsymbol{\mu})\end{array}\right.$$

$$\mathbf{F} = \{\boldsymbol{\mu} = \mu_n \mathbf{n} + \boldsymbol{\mu}_t \in \mathbb{R}^3 \ / \ |\boldsymbol{\mu}_t| \leq \mu \mu_n\}$$

$$E(\boldsymbol{\mu}) = \mu_n \left[D^k + \Delta t \, \mathbf{n} \cdot A^k \left(\frac{\mathbf{u}_{\boldsymbol{\mu}} + \bar{\mathbf{u}}^{k+1}}{2} \right) \right] + \boldsymbol{\mu}_t \cdot \left[\Delta t \, \mathrm{T}^k A^k \left(\frac{\mathbf{u}_{\boldsymbol{\mu}} + \bar{\mathbf{u}}^{k+1}}{2} \right) \right]$$

▷ Can be solved using Projected Gradient Algorithms



- On convex numerical schemes for inelastic contacts with friction, with H. Bloch, ESAIM:Proc 75 (2023), pp. 24-59
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Convex optimization for friction?

$$\left| \begin{array}{l} \lambda \in F \\ E(\lambda) = \min\limits_{\mu \in F} E(\mu) \end{array} \right.$$

Dual problem

$$\left| \begin{array}{l} \boldsymbol{\lambda} \in \mathbf{F} \\ E(\boldsymbol{\lambda}) = \min\limits_{\boldsymbol{\mu} \in \mathbf{F}} E(\boldsymbol{\mu}) \end{array} \right.$$

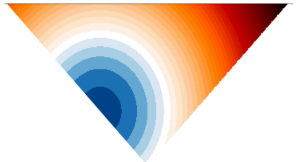
Discrete energy

$$\mathbf{F} = \{\boldsymbol{\mu} = \mu_n \mathbf{n} + \boldsymbol{\mu}_t \in \mathbb{R}^3 \ / \ |\boldsymbol{\mu}_t| \leq \mu \mu_n\}$$

Non derivable

$$E(\boldsymbol{\mu}) = \mu_n \left[D^k + \Delta t \, \mathbf{n} \cdot A^k \left(\frac{\mathbf{u}_{\boldsymbol{\mu}} + \bar{\mathbf{u}}^{k+1}}{2} \right) \right] + \boldsymbol{\mu}_t \cdot \left[\Delta t \, \mathrm{T}^k A^k \left(\frac{\mathbf{u}_{\boldsymbol{\mu}} + \bar{\mathbf{u}}^{k+1}}{2} \right) \right]$$

- Can be solved using Projected Gradient Algorithms



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Convex optimization for friction?

$$\nabla J(u) \in -\lambda \partial g[u]$$

$$g(u) \leq 0, \quad \lambda \geq 0$$

$$g(u)\lambda = 0$$

Optimality conditions

$$M \frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} = \mathbf{f}^e + A^{k,T}(\lambda_n \mathbf{n}^k + \boldsymbol{\lambda}_t) \qquad \text{(FPD)}$$

$$\lambda_n \geq 0 \qquad \text{(Norm. Cont.)}$$

$$D^k + \Delta t \, \nabla D^k \cdot \mathbf{v}^{k+1} \geq \mu \Delta t \, |\mathrm{T}^k A^k \mathbf{u}^{k+1}|$$

$$(D^k + \Delta t \, \nabla D^k \cdot \mathbf{v}^{k+1} - \mu \Delta t \, |\mathrm{T}^k A^k \mathbf{u}^{k+1}|)\lambda_n = 0$$

If $\mathrm{T}^k A^k \mathbf{u}^{k+1} = 0 \qquad |\boldsymbol{\lambda}_t| \leq \mu \lambda_n$ (Tang. Cont.)

If $\mathrm{T}^k A^k \mathbf{u}^{k+1} \neq 0 \qquad \boldsymbol{\lambda}_t = -\mu \lambda_n \dfrac{\mathrm{T}^k A^k \mathbf{u}^{k+1}}{|\mathrm{T}^k A^k \mathbf{u}^{k+1}|}$

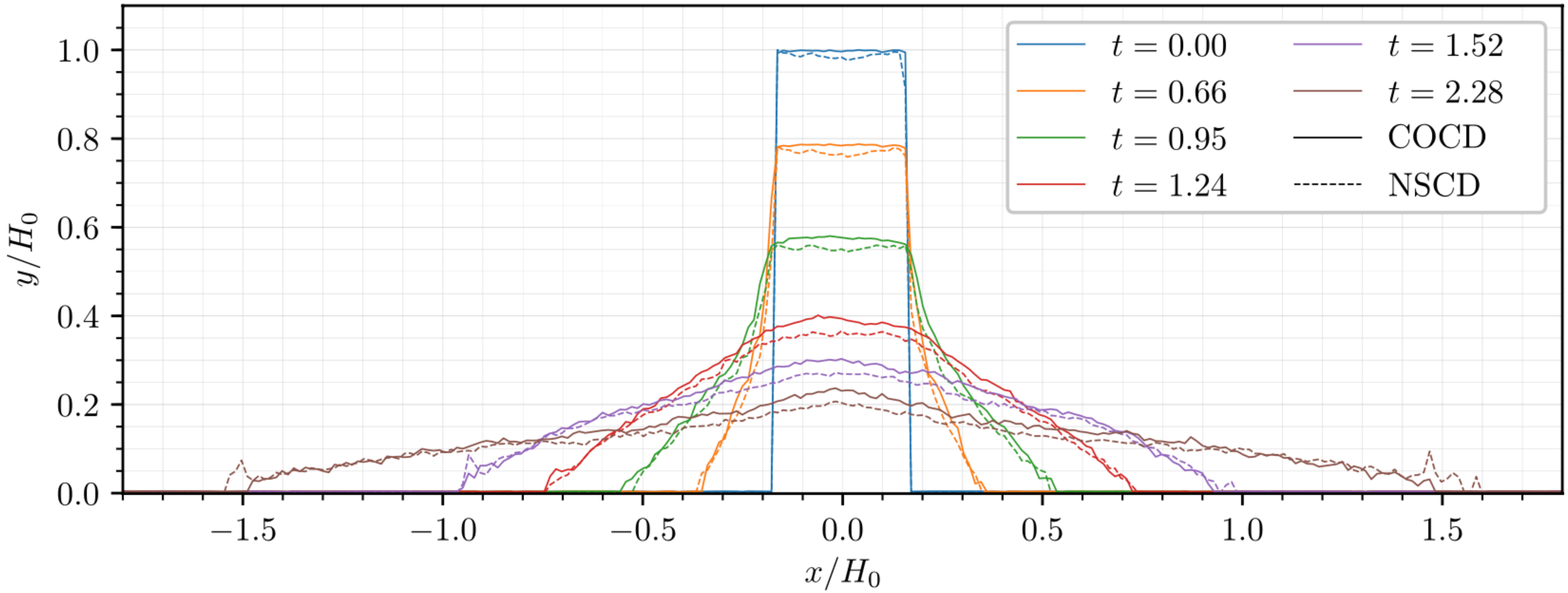
An iterative approach for cone complementarity problems for nonsmooth dynamics, Anitescu, Tasora. In: Computational Optimization and Applications 47:2 (2010).
An optimization-based model for dry granular flows: application to granular collapse on erodible beds, with H. Martin, A. Mangeney, Y. Maday, B. Maury. In: JCP 498 (2024), pp.112665.

Numerical tests.

- PhD Hugo Martin [LJLL, IPGP]
  with Y. Maday, A. Mangeney, B. Maury

- Influence of convexification?

Granular column collapse:



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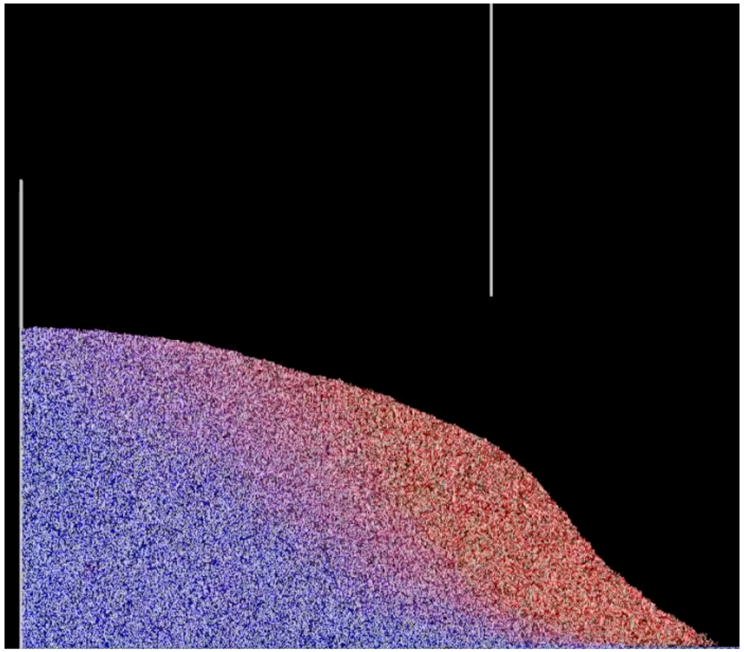

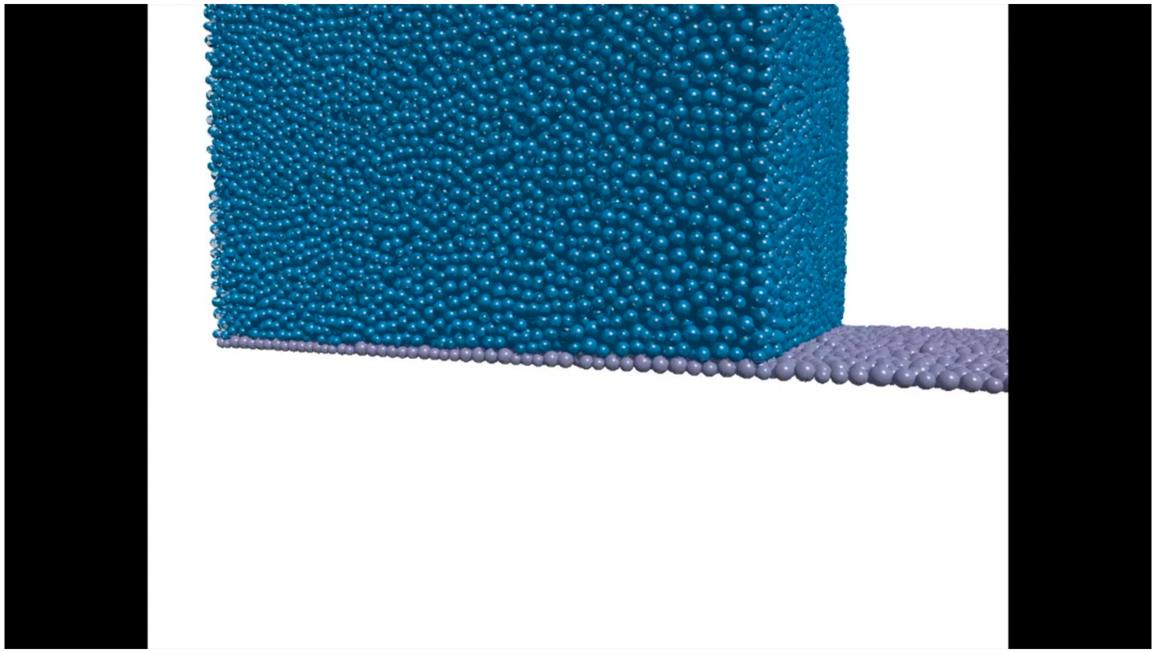

- 2D simulation - 70 000 spheres
- 3D simulation - 112 000 spheres

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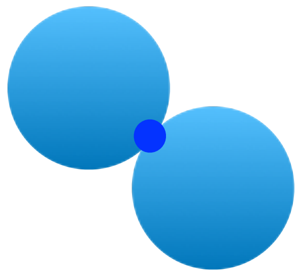
Contact law

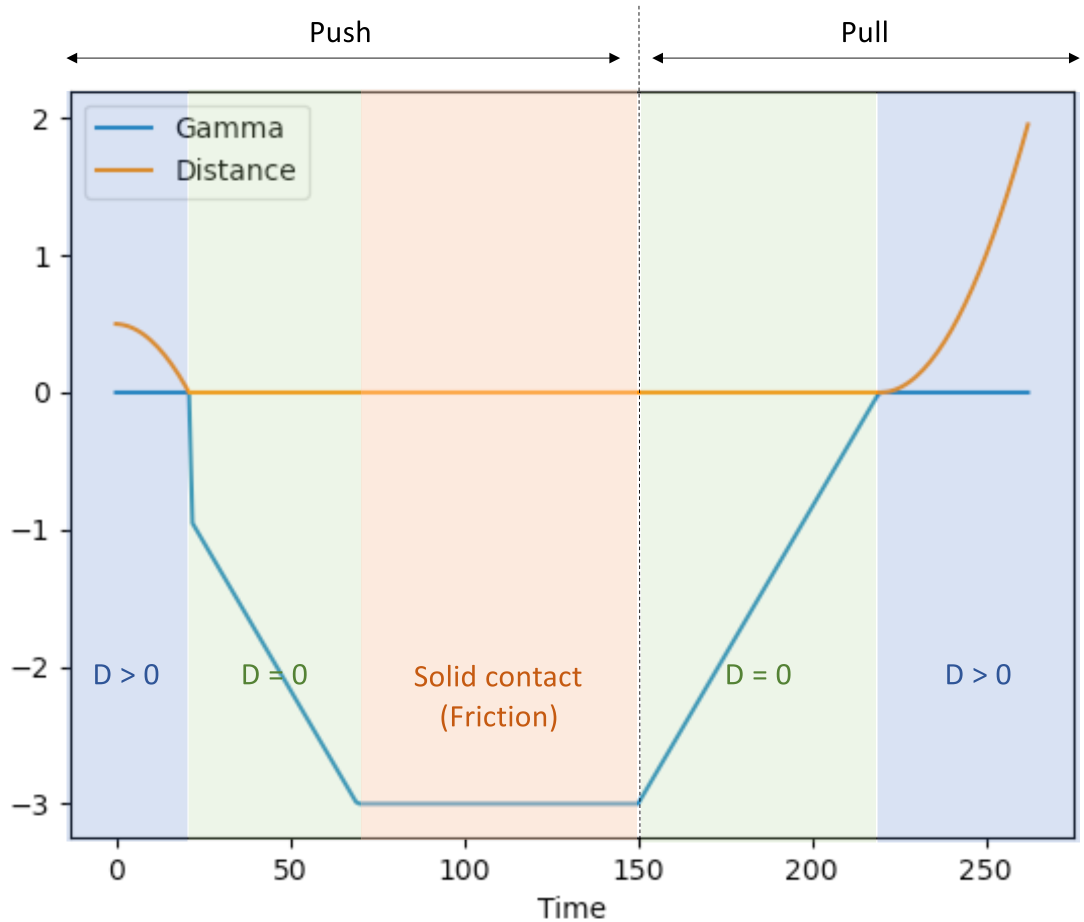


Gluey contacts
Inelastic collisions
Coulomb friction law

[PhD Quentin Houssier]

80 | Prime

Coupling lubrication and friction.



Coupling lubrication and friction.

Discrete energy [reminder]

- Gluey contact

$$(\lambda_+, \lambda_-) \in F = \mathbb{R}^+ \times \mathbb{R}^+$$

$$E(\mu_+, \mu_-) = (\mu_+ - \mu_-)\left(q^k + \Delta t\, \frac{u_\mu + \bar{u}^{k+1}}{2}\right)$$

- Friction

$$\boldsymbol{\lambda} \in \mathbf{F} = \{\boldsymbol{\mu} = \mu_n \mathbf{n} + \boldsymbol{\mu}_t \in \mathbb{R}^3 \;/\; |\boldsymbol{\mu}_t| \leq \mu \mu_n\}$$

$$E(\boldsymbol{\mu}) = \mu_n \left[D^k + \Delta t\, \mathbf{n} \cdot A^k \left(\frac{\mathbf{u}_\mu + \bar{\mathbf{u}}^{k+1}}{2}\right)\right] + \boldsymbol{\mu}_t \cdot \left[\Delta t\, \mathrm{T}^k A^k \left(\frac{\mathbf{u}_\mu + \bar{\mathbf{u}}^{k+1}}{2}\right)\right]$$

Coupling lubrication and friction.

Discrete energy

- Gluey contact and friction

$$(\boldsymbol{\lambda}, \lambda_-) \in \mathbf{F}$$

$$\mathbf{F} = \{(\boldsymbol{\mu}, \mu_-) = (\mu_n \mathbf{n} + \boldsymbol{\mu}_t, \mu_-) \in \mathbb{R}^3 \times \mathbb{R} \ / \\ \mu_- \geq 0, \ |\boldsymbol{\mu}_t| \leq \mu \mu_n, \ \mu_- \, \mu_n = 0\}$$

$$E(\boldsymbol{\mu}) = (\mu_n - \mu_-) \left[D^k + \Delta t \, \mathbf{n} \cdot A^k \left(\frac{\mathbf{u}_{\boldsymbol{\mu}} + \bar{\mathbf{u}}^{k+1}}{2} \right) \right] \\ + \boldsymbol{\mu}_t \cdot \left[\Delta t \, \mathrm{T}^k A^k \left(\frac{\mathbf{u}_{\boldsymbol{\mu}} + \bar{\mathbf{u}}^{k+1}}{2} \right) \right]$$

Coupling lubrication and friction.

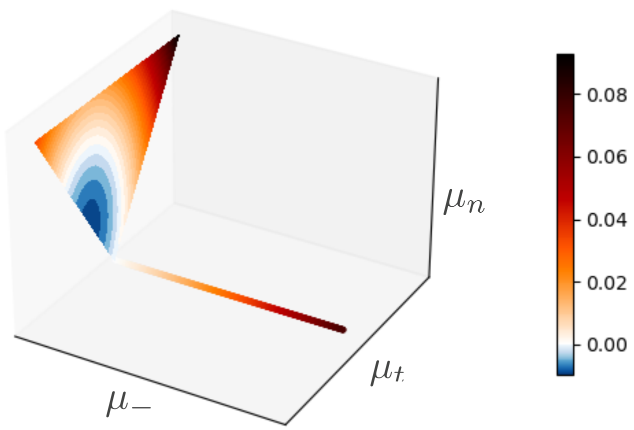
Discrete energy

- Gluey contact and friction

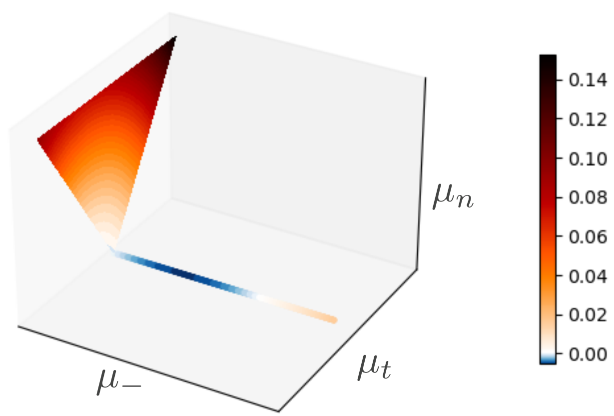
$$(\boldsymbol{\lambda}, \lambda_-) \in \mathbf{F}$$

Non convex

$$\mathbf{F} = \{(\boldsymbol{\mu}, \mu_-) = (\mu_n \mathbf{n} + \boldsymbol{\mu}_t, \mu_-) \in \mathbb{R}^3 \times \mathbb{R} \ / \ \mu_- \geq 0,\ |\boldsymbol{\mu}_t| \leq \mu \mu_n,\ \mu_- \mu_n = 0\}$$

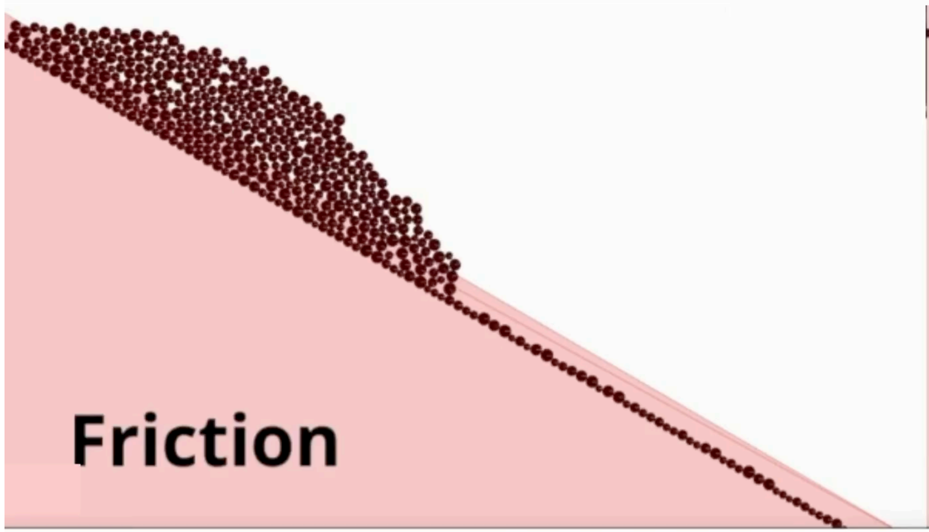


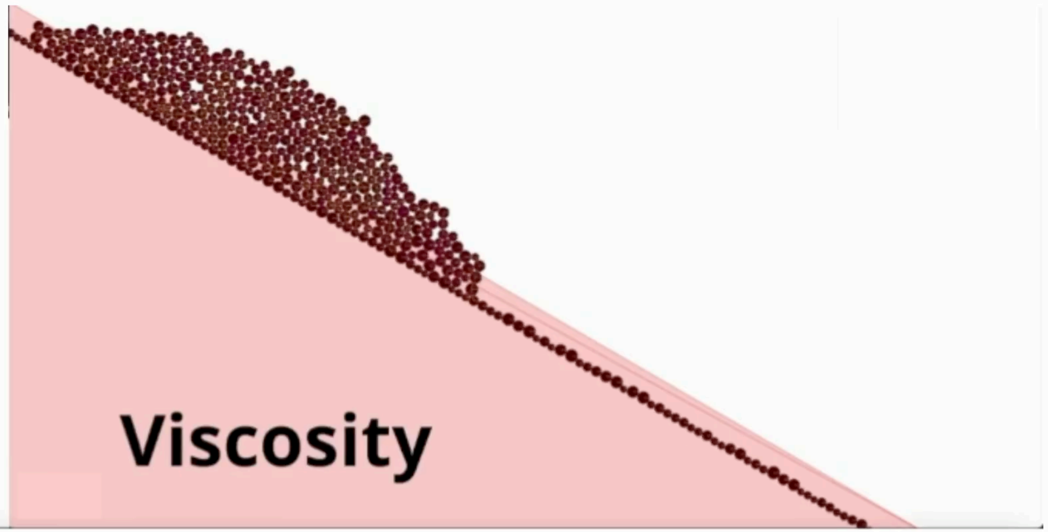
Push: activate friction



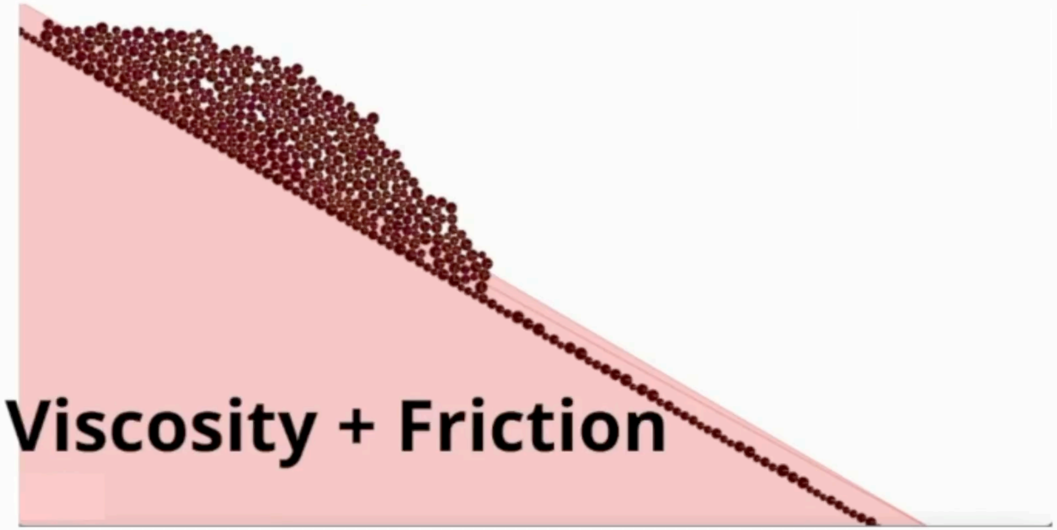
Pull: activate lubrication

Numerical tests



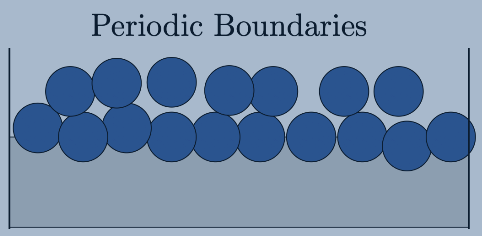


Simulations Done with Scopi

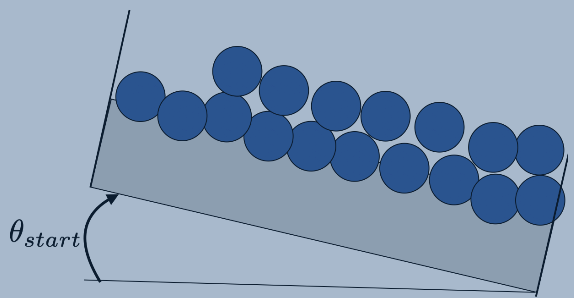


Numerical experiment: avalanche angle

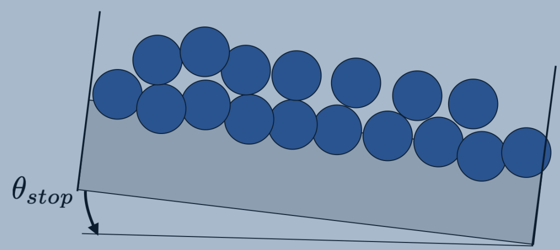
- Step 1



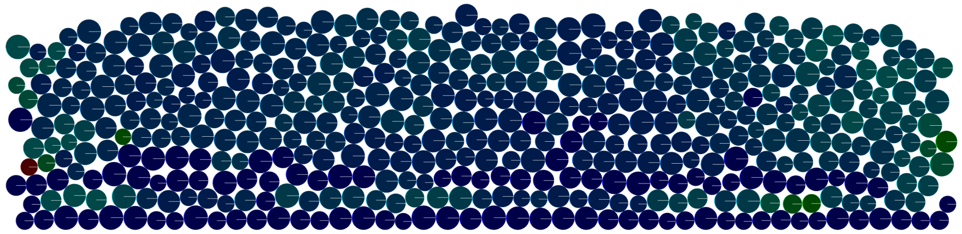
- Step 2



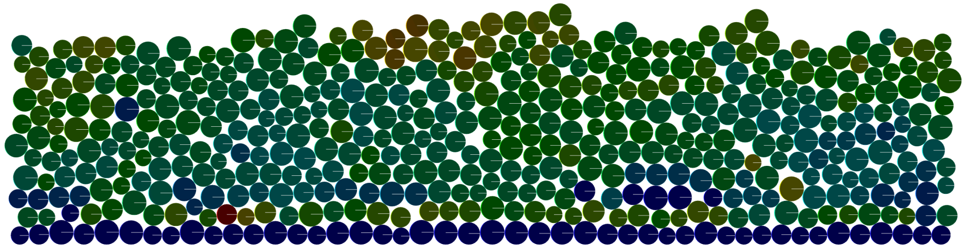
- Step 3



- Lubrication, No friction, $\theta = 8^o$, no flow



- Lubrication, No friction, $\theta = 12^o$, flowing



	θ_{start}	θ_{stop}
No friction	12	11
Friction	17	12

[with B. Darbois-Texier, G. Gauthier, FAST]

Prospects.

- Understand the « projection » on the non convex set
- Convergence study of the different schemes
- Modelling
  - Include elastic collisions
- Avalanche angle experiment:
  - Local stress, Coordination number
  - Influence of inertia
- Soils: Consider fibers, living beings
- Add various local forces (cohesion, repulsion, shear-thickening)
- Taking lubrication effect and solid collision in macroscopic PDE models for granular flows (with Charlotte Perrin)

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