Analysis of a sedimenting suspension near a vertical wall

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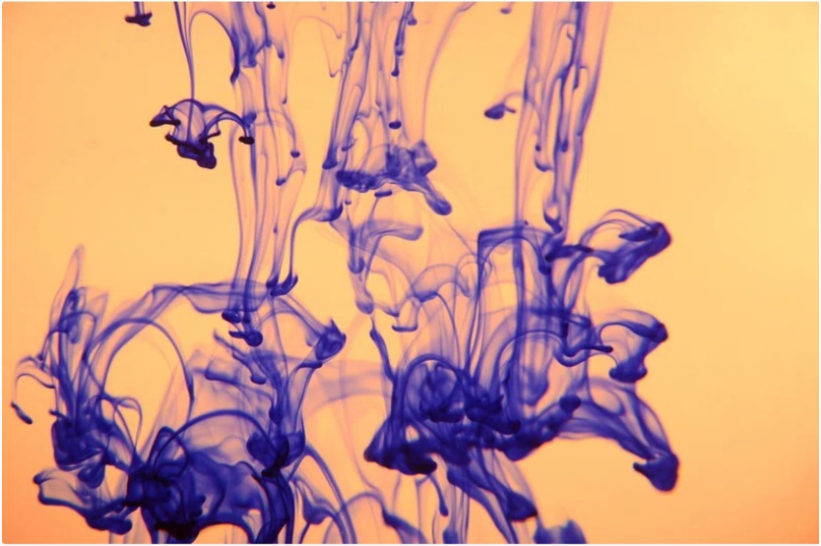
Introduction



Microscopic model of (inertialess) sedimentation

$$
\begin{aligned}
-\mu\Delta u + \nabla p = f, \quad & \text{in } \Omega \setminus (\cup_i B_i) \\
\operatorname{div} u = 0, \quad & \text{in } \Omega \setminus (\cup_i B_i) \\
u|_{\partial\Omega} = 0 & \\
u = u_i + \omega_i \times (x - X_i) \quad & \text{in } B_i, \quad \forall i = 1, \cdots, N \\
\int_{\partial B_i} \sigma_\mu(u,p) n = F, \quad \int_{\partial B_i} \sigma_\mu(u,p) n \times (X - X_i) = 0, & \quad \forall i = 1, \cdots, N
\end{aligned}
$$

$\sigma_\mu(u,p) = 2\mu D(u) - pI$ is the Newtonian stress tensor.

$F = -mge$ is the gravitational force, $e = (0,0,1)^t$,
$m = \frac{4}{3}\pi R^3(\rho_p - \rho_f)$.

n the unit normal on ∂B_i pointing inwards $B_i = B(X_i, R)$.

Literature overview

- Analysis of the steady problem

  Computation of the effective viscosity

  → Haines and Mazzucato 2012, Niethammer and Schubert 2020, Hillairet and Wu (2019), Duerinckx and Gloria 2019-2023, Gérard-Varet and co-authors (Hillairet, Höfer, Mecherbet)...

  Computation of the effective settling velocity of the particles

  → Batchelor (1972), Hasimoto (1959), Hillairet and Höfer 2023

- Coupling with the particles motion and analysis of the dynamics

  → Höfer (2018), Mecherbet (2019), Höfer and Schubert (2021, 2023), Duerinckx (2023)

Mean sedimentation speed of the particles

If $\rho^N(t=0) \underset{N\to\infty}{\rightharpoonup} \rho_0$ then $\rho = \lim_{N\to\infty} \rho^N$ solves

$$\begin{aligned} -\mu\Delta u + \nabla p = FN\rho, \quad \operatorname{div} u = 0 \\ \partial_t \rho + \operatorname{div}\left(\left(\tfrac{F}{6\pi\mu R} + u\right)\rho\right) = 0 \qquad \text{in } \mathbb{R}^3 \\ \rho(t=0) = \rho_0 \end{aligned}$$

which means that the amplitude of the mean velocity of the particles is

$$|V| \approx \left|\frac{F}{6\pi\mu R}\right| + \left|\frac{FN}{\mu L}\right| \approx |V^{St}| \max(1, NR)$$

$L = \operatorname{diam} \operatorname{supp} \rho$ the length scale of the particle cloud

$V^{St} = \frac{F}{6\pi\mu R}$ the velocity of one single particle.

→ Höfer (2018), Mecherbet (2019), Höfer and Schubert (2021, 2023)

Mean sedimentation speed of the particles

Hindered settling in the case of homogeneously distributed particles

→ Hasimoto (1959). Periodically distributed particles

$$V \approx V^{St}(1 - c_1\phi^{1/3}), \quad c_1 > 0$$

ϕ the volume fraction of the particles and $V^{St} = \frac{2}{9}R^2(\rho_p - \rho_f)g$ the sedimentation speed of one spherical particle of radius R.

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→ Batchelor (1972). Particles distributed with hardoce Poisson process with distance $2R$ in $\mathbb{R}^3$

$$V \approx V^{St}(1 - c_2\phi), \quad c_2 > 0$$

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→ Hillairet and Höfer (2023). Identification of the hindered settling for general configurations with explicit formula of the mean settling in terms of the homogeneity defect.

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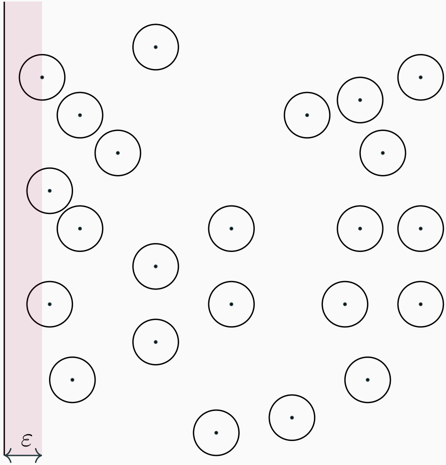
Presence of a particle depleted layer near the wall



$$\rho^N = \frac{1}{N}\sum_i \delta_{X_i}$$

Observed phenomena due to the particle depleted layer

Apparent slip

Presence of non-zero downward velocities just outside the depletion layer.

→ *Slip flow and wall depletion layer of microfibrillated cellulose suspensions in a pipe flow*. A. Koponen, S. Haavisto, J. Salmela and M. Kataja. 2019

→ *Apparent slip in colloidal suspensions*. A. Abbasi Moud, J. Piette, M. Danesh, G. C. Georgiou, S. G. Hatzikiriakos. 2022.

Observed phenomena due to the particle depleted layer

Intrinsic convection (In the case of horizontally homogeneous distribution of particles)
presence of an additional downward flow in the bulk which points upward just outside the depletion layer.

→ Mazur, Beenakker and co-workers (1985-1988), Nozières (1987)[1]

→ D. Bruneau, F. Feuillebois and co-workers (1996-1998)

→ Y. Peysson and E. Guazzelli (1997)

This effect is found to be, experimentally, much smaller than the amplitude $V^{St}\phi$ and seems to disappear with increasing concentrations.

[1] The phenomena was called *essential convection* by Beenakker and Mazur and later was called *intrinsic convection* by Nozières

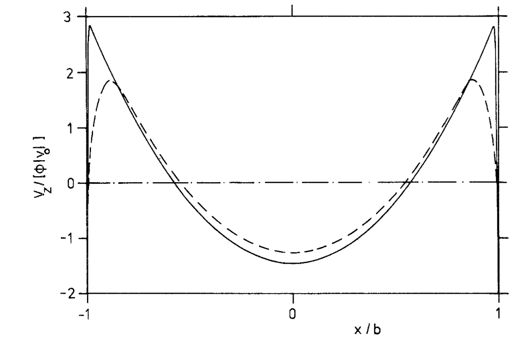


Figure 1: (Geigenmüller and Mazur, 1988): The mean volume flow due to intrinsic convection between two parallel plates. The dashed line corresponds to $b/a = 10$, the solid line to $b/a = 100$.
(b the length of the vessel, a the radius of particles)

Assumptions and main results

Nondimensionalized model

Consider the half space $\Omega^\varepsilon = (-\varepsilon, +\infty) \times \mathbb{R}^2$ where the depletion layer $D_\varepsilon = (-\varepsilon, 0) \times \mathbb{R}^2$, of width $\varepsilon > 0$, is free of particles centers. Namely,

$$X_1, \ldots, X_N \in K \Subset \overline{\Omega^0}, \quad \Omega^0 = \mathbb{R}_+^* \times \mathbb{R}^2. \tag{H1}$$

Denote by $B_i = B(X_i, R)$ the ith particle.

$$
\begin{aligned}
-\Delta u^{N,\varepsilon} + \nabla p^{N,\varepsilon} &= f, \quad \text{in } \Omega^\varepsilon \setminus (\cup B_i) \\
\operatorname{div} u^{N,\varepsilon} &= 0, \quad \text{in } \Omega^\varepsilon \setminus (\cup B_i) \\
u^{N,\varepsilon}|_{\partial\Omega^\varepsilon} &= 0 \\
D(u^{N,\varepsilon}) &= 0 \quad \text{in } B_i, \quad 1 \leq i \leq N \\
\int_{\partial B_i} \sigma(u^{N,\varepsilon}, p^{N,\varepsilon}) n &= -\frac{1}{N} e, \quad 1 \leq i \leq N \\
\int_{\partial B_i} \sigma(u^{N,\varepsilon}, p^{N,\varepsilon}) n \times (x - X_i) &= 0, \quad 1 \leq i \leq N
\end{aligned}
\qquad (E^{N,\varepsilon})
$$

with $f \in L^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$.

Assumptions

- We denote by ρ^N the empirical measure defined by

$$\rho^N = \frac{1}{N}\sum_i \delta_{X_i}$$

  and consider ρ a bounded density compactly supported in $\overline{\Omega^0}$.
- We assume that there exists $\theta > 1$ such that

$$\epsilon > \theta R \tag{H2}$$

- Denoting $d_{min} = \min_{i\neq j}|X_i - X_j|$, we assume that

$$d_{\min} \geq \max\left(\frac{C}{N^{1/3}}, 2\theta R\right) \tag{H3}$$

Approximation by a continuous model

Adapting the classical argument to the half space setting we end up with the following effective model

$$
\begin{aligned}
-\Delta u^{\varepsilon} + \nabla p^{\varepsilon} &= f - \rho e, \quad \text{in } \Omega^{\varepsilon} \\
\operatorname{div} u^{\varepsilon} &= 0, \quad \text{in } \Omega^{\varepsilon} \\
u^{\varepsilon}|_{\partial\Omega^{\varepsilon}} &= 0
\end{aligned}
\tag{E^{ε}}
$$

Theorem

Let $1 < q < 3/2$, $Q \Subset \overline{\Omega^\varepsilon}$. *There exists* $C > 0$ *depending on* q, Q *and on the constant* θ *in* (H2), *such that ,*

$$\|u^{N,\varepsilon} - u^\varepsilon\|_{L^q(Q)} \leq C\Big(\phi + \|\rho^N - \rho\|_{(W^{2,q'}(\Omega^0))^*} + R\Big)$$

where $\phi = NR^3$ *the particles volume fraction.*

Remarks on the assumptions

- The norm $\|\rho^N - \rho\|_{(W^{2,q'}(\Omega^0))^*}$ can be replaced by the first Wasserstein distance $W_1(\rho^N, \rho)$ between ρ^N and ρ and consequently by $W_p(\rho^N, \rho)$ for any $p \in [1, +\infty]$ if one assumes $\rho \in L^1(\Omega^0)$.
- The control on the minimal distance (H3) can be relaxed to $d_{\min} > 2\theta R$ together with any assumption ensuring a uniform bound on

$$\frac{1}{N}\sum_{j\neq i}\frac{1}{|X_i - X_j|^2} \leq C,$$

A crude approximation of the obtained model

$$\begin{aligned} -\Delta u^\varepsilon + \nabla p^\varepsilon &= f - \rho e, \quad \text{in } \Omega^\varepsilon \\ \operatorname{div} u^\varepsilon &= 0, \quad \text{in } \Omega^\varepsilon \\ u^\varepsilon|_{\partial\Omega^\varepsilon} &= 0 \end{aligned} \tag{E^ε}$$

would be

$$\begin{aligned} -\Delta u^0 + \nabla p^0 &= f - \rho e, \quad \text{in } \Omega^0 \\ \operatorname{div} u^0 &= 0, \quad \text{in } \Omega^0 \\ u^0|_{\partial\Omega^0} &= 0 \end{aligned} \tag{E^0}$$

where the effect of the boundary layer is neglected.

Boundary layer analysis

Theorem

Assume that f and ρ are smooth, compactly supported in $\overline{\Omega^0}$. Then, for all $m \in \mathbb{N}$, one has

$$u^\varepsilon = u^0 + \varepsilon u^1 + \cdots + \varepsilon^m u^m + O(\varepsilon^{m+1}) \quad \text{in } \dot{H}^1(\Omega^0)$$

where u^0 solves (E^0), and for each $i \geq 1$, u^i solves an homogeneous Stokes equation in Ω^0 with inhomogeneous Dirichlet data at $\partial\Omega^0$ coming from lower order profiles.

Apparent slip

Corollary

Let ρ and f as in the previous theorem, and u^S the solution of the Stokes system with Navier slip boundary condition:

$$\begin{aligned} -\Delta u^S + \nabla p^S &= f - \rho e, \quad \text{in } \Omega^0 \\ div\, u^S &= 0, \quad \text{in } \Omega^0 \\ u^S|_{\partial\Omega^0} &= \varepsilon(0, \partial_1 u_2^S, \partial_1 u_3^S)|_{\partial\Omega^0} \end{aligned} \tag{1}$$

Then, one has the estimate

$$\|u^\varepsilon - u^S\|_{\dot{H}^1(\Omega^0)} = O(\varepsilon^2)$$

Navier slip boundary condition on an example

consider the case of a shear flow driven by a constant downward pressure gradient:

$$-\Delta u + \nabla p = -e, \quad \operatorname{div} u = 0 \quad \text{ in } (0,1)\times\mathbb{R}$$

In the case of Dirichlet conditions $u|_{x_1=0} = u|_{x_1=1} = 0$, the solution is the usual Poiseuille flow

$$u^0(x_1,x_2) = \big(0, \frac{1}{2}x_1(x_1-1)\big)$$

while in the case of Navier conditions

$$u|_{x_1=0} = (0, \partial_1 u_2|_{x_1=0}), \quad u|_{x_1=1} = (0, -\partial_1 u_2|_{x_1=1})$$

we find

$$u^S(x_1,x_2) = \big(0, \frac{1}{2}x_1(x_1-1) - \frac{1}{2}\big)$$

which has a non-zero downward component at the boundary.

Recall that

$$\|u^{N,\varepsilon} - u^{\varepsilon}\|_{L^q(K)} \le C\Big(\phi + \|\rho^N - \rho\|_{(W^{2,q'}(\Omega^0))^*} + R\Big)$$

which yields

$$\|u^{N,\varepsilon} - u^{S}\|_{L^q(K)} \lesssim \varepsilon^2 + C(R + \phi + \|\rho^N - \rho\|_{(W^{2,q'}(\Omega^0))^*})$$

whereas

$$\begin{aligned}\|u^{N,\varepsilon} - u^{0}\|_{L^q(K)} &\ge \|u^{\varepsilon} - u^{0}\|_{L^q(K)} - \|u^{N,\varepsilon} - u^{\varepsilon}\|_{L^q(K)} \\ &\ge C'\varepsilon - C(R + \phi + \|\rho^N - \rho\|_{(W^{2,q'}(\Omega^0))^*})\end{aligned}$$

which means that u^S is a better approximation of $u^{N,\varepsilon}$ than u^0 if

$$R + \phi + \|\rho^N - \rho\|_{(W^{2,q'}(\Omega^0))^*} \ll \varepsilon \ll 1$$

Intrinsic convection

The intrinsic convection concerns the case of homogeneous distribution of particles i.e. $\rho = 1$ with $f = 0$ yielding (formally) a vanishing zero order term

$$u^0(x) = 0, \quad p^0(x) = -x_3$$

in the expansion. Direct computations of the first order term yield also

$$u^1 = 0, p^1 = 0$$

which means that the one has to compute the second order correction u^2 in the boundary layer analysis.

Going one step further in the boundary layer analysis, we find

$$\begin{cases} -\Delta u^2 + \nabla p^2 = 0, & \text{on } (0,+\infty) \times \mathbb{R}^2 \\ \operatorname{div} u^2 = 0, & \text{on } (0,+\infty) \times \mathbb{R}^2 \\ u^2|_{x_1=0} = (0,0,\frac{1}{2}) & \end{cases}$$

This gives

$$u^\varepsilon|_{x_1=0} \approx (0,0,\frac{1}{2}\varepsilon^2).$$

Back to dimensional variables, we find

$$u_3^\varepsilon|_{x_1=0} \approx \frac{\varepsilon^2 mgN}{2\mu L^3} = \frac{g}{2\mu}\varepsilon^2(\rho_s - \rho_f)\phi = \frac{\varepsilon^2}{R^2}\frac{9}{4}V^{st}\phi$$

where $V^{st} = \frac{9}{2}R^2\frac{(\rho_p-\rho_f)}{\mu}g$ the sedimentation velocity of one particle.

Sketch of the proof

Without loss of generality we assume that $\varepsilon = 0$.

The results adapt straightforwardly to $\Omega^\varepsilon = (-\varepsilon, +\infty) \times \mathbb{R}^2$ by translation.

Fundamental solution to the Stokes equation on the half space

Given $y \in (0, +\infty) \times \mathbb{R}^2$, we set $x \mapsto G(x, y)$ the unique solution to

$$\begin{cases} -\Delta u + \nabla q &= \delta_y \mathbb{I}, \text{ on } \Omega^0 \\ \operatorname{div}(u) &= 0, \text{ on } \Omega^0 \\ u|_{x_1=0} &= 0 \end{cases}$$

where $\mathbb{I}$ is the identity matrix in $3d$. We have for all $x \neq y$

$$|G(x, y)| \leq \frac{C}{|x - y|}, \quad |\nabla_y G(x, y)| + |\nabla_x G(x, y)| \leq \frac{C}{|x - y|^2}$$

→ Z. Gimbutas, L. Greengard, and S. Veerapaneni, Simple and efficient representations for the fundamental solutions of stokes flow in a half-space, Journal of Fluid Mechanics, 776 (2015)

Stokes flow past a sphere in the half space

We set $\mathcal{U}_r^{st}[F](\cdot, y_0) \in \dot{H}^1(\Omega^0)$ the unique solution of the Stokes equation

$$\left\{ \begin{array}{rcl} -\Delta u + \nabla q & = & 0, \text{ on } \Omega^0 \setminus \overline{B(y_0, r)} \\ \operatorname{div}(u) & = & 0, \text{ on } \Omega^0 \setminus \overline{B(y_0, r)} \\ Du & = & 0, \text{ on } B(y_0, r) \\ u|_{x_1=0} & = & 0 \end{array} \right.$$

$$\int_{\partial B(y_0,r)} \sigma(u,p)n = F, \quad \int_{\partial B(y_0,r)} [\sigma(u,p)n] \times (x - y_0) = 0$$

Approximation of the flow past a sphere in the half space

For any $r > 0$ and $y_0 \in \Omega^0$ satisfying $(y_0)_1 > \theta r$, there exists $\mathcal{H}_r[F](\cdot, y_0)$ such that for all $x \notin B(y_0, \theta r)$

$$\mathcal{U}_r^{st}[F](x, y_0) = G(x, y_0)F + \mathcal{H}_r[F](x, y_0)$$

$$|\mathcal{H}_r[F](x, y_0)| \leq r\frac{C_\theta |F|}{|x - y_0|^2}, \quad |\nabla_x \mathcal{H}_r[F](x, y_0)| \leq r\frac{C_\theta |F|}{|x - y_0|^3},$$

Assume for simplicity $f = 0$ and $\Omega^\epsilon = (0, +\infty) \times \mathbb{R}^2$. Consider the approximation

$$u_{\text{app}} = \sum_j \mathcal{U}_R^{st}[F](x, X_j)$$

with $F = -\frac{1}{N} e$. We have

$$\begin{cases} -\Delta u_{\text{app}} + \nabla p_{\text{app}} &= 0, & \text{on } \Omega^0 \setminus \overline{\bigcup B_i} \\ \operatorname{div}(u_{\text{app}}) &= 0, & \text{on } \Omega^0 \setminus \overline{\bigcup B_i} \\ D u_{\text{app}} &= \sum_{j \neq i} \mathcal{U}_R^{st}[F](x, X_j), & \text{on } \overline{\bigcup B_i} \\ u_{\text{app}}|_{x_1 = 0} &= 0 \end{cases}$$

$$\int_{\partial B_i} \sigma(u_{\text{app}}, p_{\text{app}}) n = F, \quad \int_{\partial B_i} [\sigma(u_{\text{app}}, p_{\text{app}}) n] \times (x - X_i) = 0$$

by a standard variational argument we get

$$\|D(u^{N,\varepsilon} - u_{\text{app}})\|_{L^2(\Omega^0)} \leq C \|D u_{\text{app}}\|_{L^2(\bigcup B_i)} \lesssim \phi^{1/2}$$

Setting

$$v^N = -\frac{1}{N}\sum_i G(\cdot, X_i)e$$

we have

$$\begin{cases} -\Delta v^N + \nabla q^N &= -\frac{1}{N}\sum_i \delta_{X_i} e, \quad \text{in } \Omega^0 \\ \operatorname{div} v^N &= 0, \quad \text{in } \Omega^0 \\ v^N|_{x_1=0} &= 0 \end{cases}$$

and

$$\|u_{\text{app}} - v^N\|_{L^q(K)} \leq C_K R$$

for $K \Subset \overline{\Omega^0}$ and $q < 3/2$

Hence setting u^0 as the unique solution to

$$\begin{cases} -\Delta u^0 + \nabla q^0 & = & -\rho e, \quad \text{in } \Omega^0 \\ \operatorname{div} u^0 & = & 0, \quad \text{in } \Omega^0 \\ u^0|_{x_1=0} & = & 0 \end{cases}$$

we have

$$\|v^N - u^0\|_{L^q(K)} \leq C_K \|\rho^N - \rho\|_{(W^{2,q'}(\Omega^0))^*}$$

for $K \Subset \overline{\Omega^0}$ and $q < 3/2$.

Boundary layer analysis

We aim at giving an approximation of u^ε

$$\begin{cases} -\Delta u^\varepsilon + \nabla q^\varepsilon &= f - \rho e, \quad \text{in } \Omega^\varepsilon \\ \operatorname{div} u^\varepsilon &= 0, \quad \text{in } \Omega^\varepsilon \\ u^\varepsilon|_{x_1=-\epsilon} &= 0 \end{cases}$$

We use the notation u^i for the terms of the expansion in the domain Ω^0 and boundary layer profiles $U^i = U^i(s, x_2, x_3)$ in the depleted layer where $s \in]-1, 0[$ stands for the rescaled variable x_1/ε.

$$\begin{cases} (u^\varepsilon_{app}, p^\varepsilon_{app})(x) = \sum_{i=0}^{m} \varepsilon^i (u^i, p^i)(x_1, x_2, x_3), \quad x = (x_1, x_2, x_3) \in \Omega^0 \\ (u^\varepsilon_{app}, p^\varepsilon_{app})(x) = \sum_{i=0}^{m} \varepsilon^i (U^i, P^i)(x_1/\varepsilon, x_2, x_3), \quad x = (x_1, x_2, x_3) \in \Omega^\varepsilon \setminus \Omega^0 \end{cases}$$

Plugging the boundary layer profile in the Stokes equation we get

$$
\begin{aligned}
& -\frac{1}{\varepsilon^2}\partial_s^2 U^0 - (\partial_2^2+\partial_3^2)U^0 + \begin{pmatrix} \varepsilon^{-1}\partial_s P^0 \\ \partial_2 P^0 \\ \partial_3 P^0 \end{pmatrix} \\
& -\frac{1}{\varepsilon}\partial_s^2 U^1 - \varepsilon(\partial_2^2+\partial_3^2)U^1 + \begin{pmatrix} \partial_s P^1 \\ \varepsilon\partial_2 P^1 \\ \varepsilon\partial_3 P^1 \end{pmatrix} - \partial_s^2 U^2 - \varepsilon^2(\partial_2^2+\partial_3^2)U^2 + \begin{pmatrix} \varepsilon\partial_s P^2 \\ \varepsilon^2\partial_2 P^2 \\ \varepsilon^2\partial_3 P^2 \end{pmatrix} \\
& -\varepsilon\partial_s^2 U^3 - \varepsilon^3(\partial_2^2+\partial_3^2)U^3 + \begin{pmatrix} \varepsilon^2\partial_s P^3 \\ \varepsilon^3\partial_2 P^3 \\ \varepsilon^3\partial_3 P^3 \end{pmatrix} + \cdots = 0
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{\varepsilon}\partial_s U_1^0 + \left(\partial_2 U_2^0 + \partial_3 U_3^0\right) + \partial_s U_1^1 + \varepsilon\left(\partial_2 U_2^1 + \partial_3 U_3^1\right) \\
& \quad + \varepsilon\partial_s U_1^2 + \varepsilon^2\left(\partial_2 U_2^2 + \partial_3 U_3^2\right) + \varepsilon^2\partial_s U_1^3 + \varepsilon^3\left(\partial_2 U_2^3 + \partial_3 U_3^3\right)\cdots = 0
\end{aligned}
$$

Plugging the boundary layer profile in the Stokes equation we get [2]

$$-\frac{1}{\varepsilon^2}\partial_s^2 U^0 - (\partial_2^2+\partial_3^2)U^0 + \begin{pmatrix}\varepsilon^{-1}\partial_s P^0\\ \partial_2 P^0\\ \partial_3 P^0\end{pmatrix}$$

$$-\frac{1}{\varepsilon}\partial_s^2 U^1 - \varepsilon(\partial_2^2+\partial_3^2)U^1 + \begin{pmatrix}\partial_s P^1\\ \varepsilon\partial_2 P^1\\ \varepsilon\partial_3 P^1\end{pmatrix} - \partial_s^2 U^2 - \varepsilon^2(\partial_2^2+\partial_3^2)U^2 + \begin{pmatrix}\varepsilon\partial_s P^2\\ \varepsilon^2\partial_2 P^2\\ \varepsilon^2\partial_3 P^2\end{pmatrix}$$

$$-\varepsilon\partial_s^2 U^3 - \varepsilon^3(\partial_2^2+\partial_3^2)U^3 + \begin{pmatrix}\varepsilon^2\partial_s P^3\\ \varepsilon^3\partial_2 P^3\\ \varepsilon^3\partial_3 P^3\end{pmatrix} + \cdots = 0$$

$$\frac{1}{\varepsilon}\partial_s U_1^0 + \left(\partial_2 U_2^0 + \partial_3 U_3^0\right) + \partial_s U_1^1 + \varepsilon\left(\partial_2 U_2^1 + \partial_3 U_3^1\right)$$
$$+ \varepsilon\partial_s U_1^2 + \varepsilon^2\left(\partial_2 U_2^2 + \partial_3 U_3^2\right) + \varepsilon^2\partial_s U_1^3 + \varepsilon^3\left(\partial_2 U_2^3 + \partial_3 U_3^3\right)\cdots = 0$$

[2] -: terms of order ε^{-2}, -: terms of order ε^{-1}, -: terms of order 1,-: terms of order ε

for all $i \geq 0$, for all $(s, x_2, x_3) \in (-1, 0) \times \mathbb{R}^2$:

$$\begin{aligned} -\partial_s^2 U_1^i - (\partial_2^2 + \partial_3^2) U_1^{i-2} + \partial_s P^{i-1} &= 0, \\ -\partial_s^2 U_2^i - (\partial_2^2 + \partial_3^2) U_2^{i-2} + \partial_2 P^{i-2} &= 0, \\ -\partial_s^2 U_3^i - (\partial_2^2 + \partial_3^2) U_3^{i-2} + \partial_3 P^{i-2} &= 0, \\ \partial_s U_1^i + \partial_2 U_2^{i-1} + \partial_3 U_3^{i-1} &= 0 \end{aligned}$$

with the convention $(U^i, P^i) = 0$ for $i < 0$.

The Dirichlet condition $u^\varepsilon_{app}|_{\partial\Omega^\varepsilon} = 0$ yields: for all $i \geq 0$,

$$U^i|_{s=-1} = 0.$$

Plugging the interior expansion in the Stokes equation, we find: for all $i \geq 0$, in Ω^0

$$\begin{aligned} -\Delta u^i + \nabla p^i &= \delta_{0i}\big(f - \rho e\big), \\ \operatorname{div} u^i &= 0. \end{aligned} \quad \text{on } (0, +\infty) \times \mathbb{R}^2$$

together with conditions

$$[u^\varepsilon_{app}]|_{\partial\Omega^0} \approx 0, \quad [\sigma(u^\varepsilon_{app}, p^\varepsilon_{app})n]|_{\partial\Omega^0} \approx 0$$

which reflect continuity of the velocity field and the stress tensor at the interface $\partial\Omega^0$ give for all $i \geq 0$:

$$U^i|_{s=0} = u^i|_{x_1=0}$$

$$\partial_s U^i|_{s=0} - \begin{pmatrix} P^{i-1}|_{s=0} \\ 0 \\ 0 \end{pmatrix} = \partial_1 u^{i-1}|_{x_1=0} - \begin{pmatrix} p^{i-1}|_{x_1=0} \\ 0 \\ 0 \end{pmatrix}.$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have

$$\left\{\begin{array}{rcl} \partial_s^2 U^0 & = & 0 \text{ on } (-1,0)\times\mathbb{R}^2 \\ \partial_s U_1^0 & = & 0 \text{ on } (-1,0)\times\mathbb{R}^2 \\ U^0|_{s=-1} & = & 0 \\ \partial_s U^0|_{s=0} & = & 0 \end{array}\right.$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0$,

$$\left\{\begin{array}{rcl} \partial_s^2 U^0 & = & 0 \text{ on } (-1,0)\times\mathbb{R}^2 \\ \partial_s U_1^0 & = & 0 \text{ on } (-1,0)\times\mathbb{R}^2 \\ U^0|_{s=-1} & = & 0 \\ \partial_s U^0|_{s=0} & = & 0 \end{array}\right. \quad \Rightarrow \quad U^0 = 0$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0$,

$$\left\{ \begin{array}{rcl} \left(\partial_2 U_2^0 + \partial_3 U_3^0\right) + \partial_s U_1^1 & = & 0 \text{ on } (-1,0) \times \mathbb{R}^2 \\ U^1|_{s=-1} = 0 & = & 0 \end{array} \right.$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0$, $U_1^1 = 0$,

$$\left\{ \begin{array}{rcl} \left(\partial_2 U_2^0 + \partial_3 U_3^0\right) + \partial_s U_1^1 & = & 0 \text{ on } (-1,0) \times \mathbb{R}^2 \\ U^1|_{s=-1} = 0 & = & 0 \end{array} \right. \quad \Rightarrow \quad U_1^1 = 0$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0$, $U_1^1 = 0$,

$$\begin{cases} \partial_s P^0 &= 0 \text{ on } (-1,0)\times\mathbb{R}^2 \\ P^0|_{s=-1} &= \partial_s U_1^1|_{s=-1} - \partial_1 u^0|_{x_1=0} + p^0|_{x_1=0} \end{cases}$$

and

$$\begin{cases} -\Delta u^0 + \nabla p^0 &= -\rho e, & \text{on } (0,\infty)\times\mathbb{R}^2 \\ \operatorname{div} u^0 &= 0, & \text{on } (0,+\infty)\times\mathbb{R}^2 \\ u^0|_{x_1=0} &= U^0|_{s=0} \end{cases}$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0$, $U_1^1 = 0$, $P^0 = -x_3$, $(u^0, p^0) = (0, -x_3)$,

$$\left\{ \begin{array}{rcl} \partial_s P^0 & = & 0 \text{ on } (-1,0) \times \mathbb{R}^2 \\ P^0|_{s=-1} & = & -\partial_1 u^0|_{x_1=0} + p^0|_{x_1=0} \end{array} \right. \quad \Rightarrow \quad P^0 = -x_3$$

If $\rho = 1$ then $u^0 = 0$ and $p^0 = -x_3$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0$, $U^1_1 = 0$, $P^0 = -x_3$, $(u^0, p^0) = (0, -x_3)$,

$$\left\{\begin{array}{rcl} \begin{pmatrix} \partial_s^2 U_2^1 \\ \partial_s^2 U_3^1 \end{pmatrix} & = & 0 \text{ on } (-1,0)\times\mathbb{R}^2 \\ U^1|_{s=-1} & = & 0 \\ \begin{pmatrix} \partial_s U_2^1|_{s=0} \\ \partial_s U_3^1|_{s=0} \end{pmatrix} & = & \begin{pmatrix} \partial_1 u_2^0|_{x_1=0} \\ \partial_1 u_3^0|_{x_1=0} \end{pmatrix} \end{array}\right.$$

Computation of the first order terms in the case of homogeneous distribution of particles

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Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0, \quad U^1 = 0, \quad P^0 = -x_3$, $(u^0, p^0) = (0, -x_3)$, $(u^1, p^1) = 0$,

Hence

$$\begin{cases} -\Delta u^1 + \nabla p^1 = 0, & \text{on } (0, +\infty) \times \mathbb{R}^2 \\ \operatorname{div} u^1 = 0, & \text{on } (0, +\infty) \times \mathbb{R}^2 \\ u^1|_{x_1=0} = (0, \partial_1 u_2^0, \partial_1 u_3^0)|_{x_1=0} \end{cases}$$

$$\Rightarrow (u^1, p^1) = 0$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0, \quad U^1 = 0, \quad P^0 = -x_3$, $(u^0, p^0) = (0, -x_3)$, $(u^1, p^1) = 0$,

$$\partial_s U_1^2 + \partial_2 U_2^1 + \partial_3 U_3^1 = 0, \text{ on } (-1, 0) \times \mathbb{R}^2$$
$$U_1^2|_{s=-1} = 0$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0, \quad U^1 = 0, \quad P^0 = -x_3$, $(u^0, p^0) = (0, -x_3)$, $(u^1, p^1) = 0, \; U^2_1 = 0$

$$\begin{aligned} \partial_s U^2_1 = 0, \text{ on } (-1,0)\times\mathbb{R}^2 \\ U^2_1|_{s=-1} = 0 \end{aligned} \quad \Rightarrow \quad U^2_1 = 0$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0, \quad U^1 = 0, \quad P^0 = -x_3$, $(u^0, p^0) = (0, -x_3)$, $(u^1, p^1) = 0, \; U_1^2 = 0$

$$-\partial_s^2 U_1^2 - (\partial_2^2 + \partial_3^2) U_1^0 + \partial_s P^1 = 0, \text{ on } (-1, 0) \times \mathbb{R}^2$$
$$P^1|_{s=0} = \partial_s U_1^2|_{s=0} - \partial_1 u_1^1|_{x_1=0} + p^1|_{x_1=0}$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0$, $(U^1, P^1) = 0$, $P^0 = -x_3$, $(u^0, p^0) = (0, -x_3)$, $(u^1, p^1) = 0$, $U_1^2 = 0$

$$\begin{aligned} \partial_s P^1 = 0, \text{ on } (-1,0)\times\mathbb{R}^2 \\ P^1|_{s=0} = 0 \end{aligned} \quad \Rightarrow \quad P^1 = 0$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0$, $(U^1, P^1) = 0$, $P^0 = -x_3$, $(u^0, p^0) = (0, -x_3)$, $(u^1, p^1) = 0$, $U_1^2 = 0$

$$
\left\{
\begin{array}{rcl}
-\begin{pmatrix} \partial_s^2 U_2^2 \\ \partial_s^2 U_3^2 \end{pmatrix} - \begin{pmatrix} (\partial_2^2 + \partial_3^2) U_2^0 \\ (\partial_2^2 + \partial_3^2) U_3^0 \end{pmatrix} + \begin{pmatrix} \partial_2 P^0 \\ \partial_3 P^0 \end{pmatrix} & = & 0 \text{ on } (-1, 0) \times \mathbb{R}^2 \\
U^2|_{s=-1} & = & 0 \\
\begin{pmatrix} \partial_s U_2^2|_{s=0} \\ \partial_s U_3^2|_{s=0} \end{pmatrix} & = & \begin{pmatrix} \partial_1 u_2^1|_{x_1=0} \\ \partial_1 u_3^1|_{x_1=0} \end{pmatrix}
\end{array}
\right.
$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0$, $\quad (U^1, P^1) = 0$, $P^0 = -x_3$, $(u^0, p^0) = (0, -x_3)$, $(u^1, p^1) = 0$, $\quad U_1^2 = U_2^2 = 0, U_3^2 = -\frac{1}{2}(s^2 - 1)$

$$\left\{ \begin{array}{rcl} -\begin{pmatrix} \partial_s^2 U_2^2 \\ \partial_s^2 U_3^2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} & = & 0 \text{ on } (-1,0) \times \mathbb{R}^2 \\ U^2|_{s=-1} & = & 0 \\ \begin{pmatrix} \partial_s U_2^2|_{s=0} \\ \partial_s U_3^2|_{s=0} \end{pmatrix} & = & 0 \end{array} \right.$$

$$\Rightarrow \begin{pmatrix} U_2^2 = 0 \\ U_3^2 = -\frac{1}{2}(s^2 - 1) \end{pmatrix}$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0$, $\quad (U^1, P^1) = 0$, $P^0 = -x_3$, $(u^0, p^0) = (0, -x_3)$, $(u^1, p^1) = 0$, $\quad U_1^2 = U_2^2 = 0, U_3^2 = -\frac{1}{2}(s^2 - 1)$

Finally we get

$$\begin{cases} -\Delta u^2 + \nabla p^2 & = & 0, & \text{on } (0, +\infty) \times \mathbb{R}^2 \\ \operatorname{div} u^2 & = & 0, & \text{on } (0, +\infty) \times \mathbb{R}^2 \\ u^2|_{x_1=0} & = & U^2|_{s=0} & \end{cases}$$

Computation of the first order terms in the case of homogeneous distribution of particles

We have $U^0 = 0, \quad (U^1, P^1) = 0$, $P^0 = -x_3$, $(u^0, p^0) = (0, -x_3)$, $(u^1, p^1) = 0, \quad U_1^2 = U_2^2 = 0, U_3^2 = -\frac{1}{2}(s^2 - 1)$

Finally we get

$$\begin{cases} -\Delta u^2 + \nabla p^2 &= 0, & \text{on } (0, +\infty) \times \mathbb{R}^2 \\ \operatorname{div} u^2 &= 0, & \text{on } (0, +\infty) \times \mathbb{R}^2 \\ u^2|_{x_1=0} &= U^2|_{s=0} & \end{cases}$$

which yields

$$u^2|_{x_1=0} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Thank you for your attention !